A

COMPLETE

ALGEBRA

FOR

SCHOOLS AND COLLEGES,

BY

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PREFACE.

In preparing an Algebra, complete in one book, it has been the aim of the author to render the work sufficiently elementary for beginners who have a practical knowledge of Arithmetic, and sufficiently advanced for those who intend to pursue the Higher Mathematics.

The work has been called out by a recognized want: One book embracing the essential principles of the science, clearly explained and illustrated by appropriate examples. In this way, it is hoped that the multiplication of classes may be avoided. Indeed, there seems to be no reason why two books should be required for Algebra, except perhaps the immaturity of some who make haste to begin the study.

In the preparation of the work, care has been taken to render the demonstrations clear, and to illustrate the principles by their application to a variety of carefully graded examples.

Theorems and Factoring, subjects intimately related, are treated in immediate connection.

In Equations, each numerical problem is immediately followed by a general one of like nature, the solution of which gives a formula from which the answer to the numerical problem may be deduced. The student is thus early trained to generalize and to apply formulas.

In the solutions, the equations have been numbered and the operations concisely indicated, thus affording models to guide

the student in giving to his own solutions the form suitable for publication.

The Problem of the Lights has been generalized, and all the equally illuminated points have been found, which will add greatly to the interest of the discussion.

By the method of proof known as Mathematical Induction, the Binomial Theorem is first demonstrated when the exponent is a positive integer; afterwards, by the principle of Indeterminate Coefficients, the theorem is established for any exponent.

Ratio, Proportion, Variation, the Progressions, Logarithms, and the Theory of Equations, have all received careful attention.

The proposition, *Every equation has a root*, which is usually assumed, has been rigorously demonstrated.

The beauty of the type and the arrangement of the matter will add much to the pleasure of the study.

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ALGEBRA.

1. Definitions.

1. ALGEBRA is that branch of Mathematics which treats of the general relations of quantities by means of symbols.

2. The symbols employed in Algebra are the figures, letters, and signs used in expressing quantities, operations, and relations.

3. The symbols of quantity are figures and letters.

1st. *Known quantities* are usually denoted by figures, 1, 2, 3, . . . , or by the first letters of the alphabet, a, b, c, . . .

2d. *Unknown quantities* are usually denoted by the final letters, . . . x, y, z.

3d. The symbol, 0, called *zero,* denotes the absence of quantity, or a quantity less than any assignable quantity.

4th. The symbol, ∞, called *infinity,* denotes a quantity without limit, or a quantity greater than any assignable quantity.

5th. The symbols, $'$, $''$, $'''$, . . . , called *accents,* and the symbols, $_1$, $_2$, $_3$, . . . $_n$, called *subscripts,* are attached to letters to denote quantities symmetrically arranged, or unknown quantities whose values have been determined.

The expressions, a', a'', a''', . . ., are respectively read, a prime, a second, a third, . . ., and a_1, a_2, a_3, . . . a_n, are respectively read, a one, a two, a three, . . ., a n.

4. The symbols of operation are the signs of addition, subtraction, multiplication, division, involution, and evolution.

1st. *The sign of addition* is the vertical cross, $+$, called *plus.*

Thus, $a + b$, read a plus b, indicates that b is to be added to a.

2d. *The sign of subtraction* is a short horizontal line, $-$, called *minus.*

Thus, $a - b$, read a minus b, indicates that b is to be subtracted from a.

3d. *The sign of multiplication* is the oblique cross, $\times$, or the dot, [.].

Thus $a \times b$, or $a \, . \, b$, read a multiplied by b, or a times b, or a into b, denotes the product of a and b.

If either factor is denoted by a letter, the sign of multiplication is usually omitted. Thus, $5\,a$, denotes the same as $5 \times a$, or $5 \, . \, a$, and $a\,b$ denotes the same as $a \times b$, or $a \, . \, b$.

4th. *The signs of division* are, $\div$, :, —,), $\underline{|\quad}$.

Thus, each of the expressions, $a \div b$, $a : b$, $\frac{a}{b}$, $b)a$, $a\underline{|b}$, denotes that a is to be divided by b.

5th. *The sign of involution* is a small figure or letter, called an exponent, placed at the right and a little above a symbol of quantity. Thus, $^{1,\ 2,\ 3,\ \ldots\ n}$, are signs of involution, and the expression a^1, a^2, a^3, . . . a^n, denote, respectively, the first, second, third, ... and *nth* powers of a. The exponent, ^{1}, is usually omitted; thus a^1 denotes the same quantity as a.

6th. *The sign of evolution* is the symbol, $\sqrt{\ }$, called the radical sign, or the fractional exponent, $^{\frac{1}{2}}, ^{\frac{1}{3}}, \ldots ^{\frac{1}{n}}$.

The degree of the root to be extracted is indicated by the index of the radical, a small figure or letter placed in the angle of the sign, or by the denominator of the fractional exponent.

Thus, $\sqrt[2]{a}$, $\sqrt[3]{a}$, . . . $\sqrt[n]{a}$, or $a^{\frac{1}{2}}, a^{\frac{1}{3}}, \ldots a^{\frac{1}{n}}$, denote, respectively, the square root of a, the cube root of a, ... the n^{th} root of a.

The index, ^{2}, is usually omitted. Thus, $\sqrt[2]{a}$ denotes the same as $\sqrt{a}$.

5. The symbols of relation are the signs of equality, inequality, ratio, proportion, variation, aggregation, continuation, and deduction.

1st. *The sign of equality* is two equal lines, parallel and horizontal, $=$.

Thus, $x = a$ is read x is equal to a.

2d. *The sign of inequality* is an angle placed thus $>$ or thus $<$.

Thus, $x > a$ is read x is greater than a, and $x < b$ is read x is less than b.

3d. *The sign of ratio* is the colon, [:]. Thus, $a : b$ denotes the ratio of a to b, and is equivalent to $a \div b$, or to $\frac{a}{b}$.

4th. *The sign of proportion* is the colon, double colon, and colon, : :: :. Thus, $a : b :: c : d$, read a is to b as c is to d, is a proportion, and denotes that the ratio of a to b is equal to the ratio of c to d, and is thus equivalent to the equation, $a \div b = c \div d$, or to $\frac{a}{b} = \frac{c}{d}$.

5th. *The sign of variation* is the symbol, $\propto$. Thus, $a \propto b$, denotes that a varies as b.

6th. *The signs of aggregation* are the bar, |, the vinculum, ——, the parenthesis, ( ), the bracket, [ ], and the brace, { }.

Thus, each of the expressions, $\left.\begin{matrix} a \\ +b \end{matrix}\right|$, $\overline{a+b}$, $(a+b)$, $[a+b]$, $\{a+b\}$, denotes that $a+b$ is to be treated as one quantity.

If but one sign of aggregation is required, ( ) is commonly used; if two, first ( ), then [ ]; if three, first ( ), then [ ], then { }; if more, they are repeated in the same order. Thus, $(a+b)c$, $[(a+b)c+d]e$, $\{[(a+b)c+d]e+f\}g$.

7th. *The signs of continuation* are dots, . . ., or dashes, – – –, or etc. Thus, a, a^2, a^3, . . ., or a, a^2, a^3, – – –, or a, a^2, a^3, etc., read a, a^2, a^3, and so on.

8th. *The sign of deduction* is three dots placed thus, $\therefore$, and read therefore, hence, or consequently.

9th. *The sign of reason* is three dots placed thus, $\because$, and read because or since.

2. Summary of Algebraic Symbols.

1. *Symbols of Quantity.*

1st. Of known quantities, $\begin{cases} \text{Figures, } 1,\ 2,\ 3, \ldots \\ \text{First letters, } a,\ b,\ c, \ldots \end{cases}$

2d. Of unknown quantities—Final letters, . . . x, y, z.

3d. Of limits of quantity, $\begin{cases} \text{Positive limits} \begin{cases} 0. \\ +\infty. \end{cases} \\ \text{Negative limits} \begin{cases} 0. \\ -\infty. \end{cases} \end{cases}$

4th. Of symmetrical or determined quantities, $\begin{cases} a',\ a'',\ a''', \ldots \\ a_1,\ a_2,\ a_3, \ldots \end{cases}$

2. *Symbols of Operation.*

1st. The sign of addition, $+$.

2d. The sign of subtraction, $-$.

3d. The signs of multiplication, $\times$, $\cdot$.

4th. The signs of division, $\div$, :, $-$,), $\underline{|\quad}$.

5th. The signs of involution, $^{1,\ 2,\ 3,\ \ldots\ n}$.

6th. The signs of evolution, $\begin{cases} \sqrt{\ },\ \sqrt[3]{\ },\ \sqrt[4]{\ },\ \ldots\ \sqrt[n]{\ }. \\ ^{\frac{1}{2}},\ ^{\frac{1}{3}},\ ^{\frac{1}{4}},\ \ldots,\ ^{\frac{1}{n}}. \end{cases}$

3. *Symbols of Relation.*

1st. The sign of equality, $=$.

2d. The signs of inequality, $>$, $<$.

3d. The sign of ratio, : .

4th. The sign of proportion, : :: : .

5th. The sign of variation, $\propto$.

6th. The signs of aggregation, $\begin{cases} \text{The bar, } |. \\ \text{The vinculum, } \overline{\quad\quad}. \\ \text{The parenthesis, } (\ ). \\ \text{The bracket, } [\ ]. \\ \text{The brace, } \{\ \}. \end{cases}$

7th. The signs of continuation, $\begin{cases} \cdot\ \cdot\ \cdot\ \cdot \\ -\ -\ -. \\ \text{etc.} \end{cases}$

8th. The sign of deduction, $\therefore$.

9th. The sign of reason, $\because$.

3. Definitions Continued.

1. A co-efficient of a quantity is a factor showing how many times that quantity is taken. Thus, in the expression,

$5x$, 5 is the co-efficient of x; also, in the expression, ax, a is the co-efficient of x, or 1 understood, the co-efficient of ax. The co-efficient equals the number of additions plus 1. Thus, $3x = x + x + x$, and $ax = x + x + x + \ldots$ to a terms.

2. An exponent of a quantity is a small figure or letter placed at the right, and a little above the symbol of the quantity, denoting how many times that quantity is to be taken as a factor. Thus, a^3, $(x + y)^n$, denote, respectively, the cube of a, the n^{th} power of $x + y$.

The exponent equals the number of multiplications plus 1. Thus, $a^3 = a \times a \times a$, and $a^n = a \times a \times a \ldots$ to n factors.

3. A power of a quantity is the product of factors each equal to the given quantity. Thus, $a^3 = a \times a \times a$, is the third power or cube of a, and a^n is the n^{th} power of a.

4. A root of a quantity is one of the equal factors of that quantity. Thus, a is the cube root of a^3, and x is the n^{th} root of x^n.

5. The reciprocal of a quantity is unity divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a.

6. A term is an expression not connected with any other by the sign $+$ or $-$, or it is any one of the parts of an expression which are so connected. Thus, $6xy$ is a term, and $3a^2$, $-6ab$ and $+3c$ are the terms of the expression, $3a^2 - 6ab + 3c$.

1st. *A positive term* is one which has the sign $+$, expressed or understood. Thus, $+3ax$ and $7ab$ are positive terms.

2d. *A negative term* is one which has the sign $-$. Thus, $-7xy$ is a negative term.

3d. *A simple term* is a single expression, not having parts connected by the signs $+$ or $-$. Thus, $5ax$ is a simple term.

4th. *A compound term* is a collection of terms represented as one quantity by one or more of the signs of aggregation. Thus, $[(m+n)p+r]s$ is a compound term.

5th. *A numerical term* is a number. Thus, 4, \$15 are numerical terms.

6th. *A literal term* is one containing one or more letters. Thus, ab, $7x^3$, $-6a^3x$ are literal terms.

7th. *Similar terms* are abstract numbers, similar denominate numbers, or literal terms containing the same letters, each affected with the same exponent. Thus, 4, 5; \$7, $-$\$5; $5a^3b^2c$, $-7a^3b^2c$.

8th. *The degree* of a term is the number of its literal factors, and is determined by finding the sum of the exponents. Thus, $5a^3b^2c$ is of the sixth degree, since $3+2+1=6$.

9th. *Homogeneous terms* are those of the same degree. Thus, $3a^2b$, and $-6ab^2$ are homogeneous terms.

10th. *A monomial* is a single term, simple or compound. Thus, $7a^3b$, and $8(a+b)$ are monomials.

11th. *A polynomial* is an expression of two or more terms not represented as one quantity by a sign of aggregation. Thus, $7a+5c-6d$ is a polynomial.

12th. *A binomial* is a polynomial of two terms. Thus, $a+b$ is a binomial.

13th. *A trinomial* is a polynomial of three terms. Thus, $a+b-c$ is a trinomial.

The terms of a polynomial may be written in any order, if the sign of each term be retained. Thus, $a-b+c=$ $-b+c+a=$, etc.

7. A proposition is something proposed or stated.

Propositions are definitions, axioms, postulates, theorems, problems, lemmas, corollaries, or scholiums.

1st. *A definition* is such a description of an object as will distinguish it from all other objects.

2d. *An axiom* is a self-evident truth.

3d. *An absurdity* is a self-evident falsity.

4th. *A postulate* is a self-evident possibility.

5th. *A theorem* is a truth which requires demonstration.

6th. *A problem* is something proposed for solution.

7th. *A lemma* is an auxiliary theorem or problem.

8th. *A corollary* is an obvious consequence.

9th. *A scholium* is a note or remark.

10th. *An hypothesis* is something assumed.

11th. *A formula* is a theorem expressed in algebraic language.

8. A demonstration is the proof of a proposition.

Demonstration is direct or indirect.

1st. *A direct demonstration* is one which commences with known truths and combines them in a logical process till it establishes the proposition to be proved.

2d. *An indirect demonstration*, called also the *reductio ad absurdum*, proves a proposition true by showing that the supposition that it is false involves an absurdity.

4. Axioms.

1. If equals be added to equals, the sums will be equal.
2. If equals be subtracted from equals, the remainders will be equal.
3. If equals be multiplied by equals, the products will be equal.

4. If equals be divided by equals, the quotients will be equal.

5. The same powers of equals are equal.

6. The same roots of equals are equal.

7. The whole is greater than a part.

8. The whole is equal to the sum of all the parts.

9. Things equal to the same thing are equal to each other.

5. Postulates.

1. Quantities of the same kind can be added together.

2. One quantity can be subtracted from another of the same kind.

3. A quantity can be taken any number of times.

4. The number of times that one quantity is contained in another of the same kind can be found.

5. A quantity can be divided into any number of equal parts.

6. The ratio of two quantities of the same kind can be found.

7. Any power of a quantity can be found.

8. Any root of a quantity can be found.

6. Exercises in Algebraic Notation.

1. Write the sum of x and y.

2. Write the sum of x and y multiplied by their difference.

3. Write the sum of x and y divided by their difference.

4. Write $a + b$ divided by c.

5. Write $a, + b$ divided by c.

6. Write the square root of $x + y$.

7. Write the square root of $x, + y$.

8. Write $a, + b$ multiplied by $a, - b$.

9. Write $a + b$ multiplied by the square root of $a - b$.

10. Write $a, + b$ multiplied by the square root of $a, - b$.
11. Write a homogeneous binomial.

7. Exercises in Finding Numerical Values.

Find the numerical value of the following expressions, if $a = 2,\ b = 3,\ c = 4,\ d = 5,\ x = 8,\ y = 6$.

1. $a + b - c$. *Ans.* 1.
2. $(x + y)(x - y)$. *Ans.* 28.
3. $(x + y)\,x - y$. *Ans.* 106.
4. $x + xy - y$. *Ans.* 50.
5. $[(a + b)\,c + d]\,(x - y)$. *Ans.* 50.
6. $\{[(a + b)\,c - d]\,x + y\}\,y$. *Ans.* 756.
7. $\sqrt{a^2 + d} \times \sqrt{x^2 + y^2}$. *Ans.* 30.
8. $\sqrt{(a + b)\sqrt{(x + y)(x - y)} - b}$. *Ans.* 5.
9. $a + b\sqrt{(a + d)\sqrt{xy - c + d}}$. *Ans.* 23.

ADDITION.

8. Definitions.

1. The sum of two or more quantities is their simplest expression.

2. Addition is the process of finding the sum of two or more quantities.

9. Illustrations.

1. Find the sum of $3a^2b$ and $5a^2b$.

OPERATION.

$$\begin{array}{l} 3a^2b \\ \underline{5a^2b} \\ 8a^2b \end{array}$$

3 times any quantity + 5 times the same quantity = 8 times that quantity.

2. Find the sum of $8ab^2$ and $-5ab^2$.

OPERATION.

$$\begin{array}{r} 8ab^2 \\ -5ab^2 \\ \hline 3ab^2 \end{array}$$

8 times any quantity $-$ 5 times the same quantity $=$ 3 times that quantity.

3. Find the sum of $-9mn$ and $4mn$.

OPERATION.

$$\begin{array}{r} -9mn \\ 4mn \\ \hline -5mn \end{array}$$

$-$ 9 times any quautity $+$ 4 times the same quantity $= -$ 5 times that quantity.

4. Find the sum of ax and bx.

OPERATION.

$$\begin{array}{r} ax \\ bx \\ \hline (a+b)\,x \end{array}$$

a times $x + b$ times $x = (a+b)$ times x. a and b are regarded as co-efficients of x.

5. Find the sum of $6a$, $8a$, $-5a$, and $-6a$.

OPERATION.

$$\begin{array}{r} 6a \\ 8a \\ -5a \\ -6a \\ \hline 3a \end{array}$$

$6a + 8a = 14a$. $-5a - 6a = -11a$, and $14a - 11a = 3a$.

6. Find the sum of a, $-b$, and c.

OPERATION.

$a - b + c$ Connect the quantities by their signs.

7. Find the sum of $2ab + 3xy$, $4ab + 2xy$, and $ab - 8xy + 3p$.

OPERATION.

$$\begin{array}{r} 2ab + 3xy \\ 4ab + 2xy \\ ab - 8xy + 3p \\ \hline 7ab - 3xy + 3p \end{array}$$

10. Rule.

1. *Write the quantities to be added so that similar terms, if any, shall stand in the same column.*

2. *If all the terms of a column have like signs, find the sum of their co-efficients, prefix the common sign, and annex the common literal part.*

3. *If all the terms of a column have not like signs, find the sum of the co-efficients of the positive terms, also the sum of the co-efficients of the negative terms, take their numerical difference, prefix the sign of the greater, and annex the common literal part.*

4. *Bring down, with their proper signs, the terms, if any, which are not similar to any of the others.*

11. Examples.

1. Add $3ab + 5a^2b$, $6ab - 8a^2b$, $8ab - a^2b$, and $3ab + 2a^2b$. *Ans.* $20ab - 2a^2b$.

2. Add $5xy + 4x^2z - 3xz^2$, $7xy + 9x^2z - xz^2$, and $-10xy + 20x^2z - 15xz^2$. *Ans.* $2xy + 33x^2z - 19xz^2$.

3. Add $10x^2y - 10xy^2$, $15x^2y - 5xy^2$, $x^2y + xy^2$, and $-2x^2y + 5xy^2$. *Ans.* $24x^2y - 9xy^2$.

4. Add $3a + b - c$, $9a - 6b - 10c + f$, $8a + 6b - 8c + 3f$, $4a - b + c + g$, and $6a - 3b - 8c$.
Ans. $30a - 3b - 26c + 4f + g$.

5. Add $a + b$, $c - d$, $e + f - g$.
Ans. $a + b + c - d + e + f - g$.

6. Add $5(a + b) - 6(x - y)$, $10(a + b) - 8(x - y)$, and $20(a + b) - 10(x - y)$.
Ans. $35(a + b) - 24(x - y)$.

7. Add $8a^3b + 6ab^3$, $10a^3b - 18ab^3$, $4a^3b - 20ab^3$, $9a^3b - 8ab^3$, and $-6a^3b + 15ab^3 + c + h$.
Ans. $25a^3b - 25ab^3 + c + h$.

8. Add ax, bx, cx, and dx. *Ans.* $(a+b+c+d)\,x$.

9. Add $a\,(x+y)$ and $b\,(x+y)$.
Ans. $(a+b)\,(x+y)$.

10. Add $ax+by$, $cx+dy$, $ex+fy$, and $gx+hy$.
Ans. $(a+c+e+g)\,x+(b+d+f+h)\,y$.

11. Add $5\sqrt{a+b}$, $6\sqrt{a+b}$, $7\sqrt{a+b}$, $-10\sqrt{a+b}$, and $18\sqrt{a+b}$. *Ans.* $26\sqrt{a+b}$.

12. Add $a\sqrt{x+y}$, $b\sqrt{x+y}$, $c\sqrt{x+y}$, and $d\sqrt{x+y}$.
Ans. $(a+b+c+d)\sqrt{x+y}$.

13. Add $am+bn$ and $bm+an$.
Ans. $(a+b)\,m+(a+b)\,n$.

14. Reduce $(a+b)\,m+(a+b)\,n$, by regarding m and n as co-efficients of $(a+b)$. *Ans.* $(a+b)\,(m+n)$.

15. Add $ax+by+cz$, $bx+cy+az$, and $cx+ay+bz$.
Ans. $(a+b+c)\,(x+y+z)$.

16. Add $abx^n+acy^n+bcz^n$, $acx^n+bcy^n+abz^n$, $bcx^n+aby^n+acz^n$, $2abx^n+2acy^n+2bcz^n$, $2acx^n+2bcy^n+2abz^n$, $2bcx^n+2aby^n+2acz^n$.
Ans. $3\,(ab+ac+bc)\,(x^n+y^n+z^n)$.

SUBTRACTION.

12. Definitions.

1. Subtraction is the process of taking one quantity from another.

2. The quantities considered are the subtrahend, the minuend, and the difference.

1st. *The subtrahend* is the quantity to be subtracted.

2d. *The minuend* is the quantity from which the subtrahend is to be subtracted.

3d. *The difference* is the quantity which added to the subtrahend will give the minuend.

13. Illustrations.

1. From a subtract b.

OPERATION.

$$\begin{array}{r} a \\ \underline{b} \\ a-b \end{array}$$

Since the terms are dissimilar, the subtraction can only be indicated.

PROOF.

$$\begin{array}{r} b \\ \underline{a-b} \\ a \end{array}$$

Adding the difference, $a-b$, to the subtrahend, b, the sum is the minuend, a.

2. From $a-b$ subtract $c-d$.

OPERATION.

$$\begin{array}{l} a-b \\ \underline{c-d} \\ a-b-c+d \end{array}$$

Subtracting c from $a-b$, we have $a-b-c$; but we have subtracted too much by d; $\therefore$ the remainder is too small by d; $\therefore$ $a-b-c+d$ is the true remainder. The signs of the minuend are retained, but those of the subtrahend are changed.

PROOF.

$$\begin{array}{r} c-d \\ \underline{a-b-c+d} \\ a-b \end{array}$$

Adding the difference, $a-b-c+d$, to the subtrahend, $c-d$, the sum is the minuend, $a-b$.

Hence, for all values of a, b, c, d we have the formula

$$(1)\quad a-b-(c-d)=a-b-c+d.$$

If $\left\{\begin{array}{l} b=0 \\ c=0 \end{array}\right\}$ (1) becomes (2) $a-(-d)=a+d$.

3. From $5a - 6b$ subtract $3a - 2b - c$.

OPERATION.

$$\begin{array}{l} 5a - 6b \\ \underline{3a - 2b - c} \\ 2a - 4b + c \end{array}$$

Conceiving the signs of the subtrahend changed, we have $5a - 3a = 2a$, $-6b + 2b = -4b$, and $+c$.

14. Rule.

1. *Write the subtrahend under the minuend so that similar terms, if any, shall stand in the same column.*

2. *Conceive the signs of all the terms of the subtrahend changed, and proceed as in addition.*

15. Examples.

1. From $3ax - 4by + 3cz$ take $ax + 3by - 5cz$.
Ans. $2ax - 7by + 8cz$.

2. From $10a^2b + 15b^2c - 8c^2d$ take $5a^2b - 10b^2c + c^2d$.
Ans. $5a^2b + 25b^2c - 9c^2d$.

3. From $5x^2y - 6xy^2 + 3y^2z - 5yz^2$ subtract $-4x^2y + 5xy^2 + 5y^2z - 6yz^2 - m + n$.
Ans. $9x^2y - 11xy^2 - 2y^2z + yz^2 + m - n$.

4. From $a^2 + 2ab + b^2$ take $a^2 - 2ab + b^2$.
Ans. $4ab$.

5. From $6a^2b^3c - 7abc^3 + 9ac$ take $4a^2b^3c + 3abc^3 - 8ac$.
Ans. $2a^2b^3c - 10abc^3 + 17ac$.

6. From $5a^m - 6b^m$ take $a^m + b^m - c^n$.
Ans. $4a^m - 7b^m + c^n$.

7. From x^my^n take $x^my^n - y^mz^n$.
Ans. y^mz^n.

8. From $ax + by$ take $cx + dy$.
Ans. $(a - c)x + (b - d)y$.

9. From $ax + ay + az$ take $bx + by + bz$.
Ans. $(a - b)(x + y + z)$.

10. From $(a+b)\sqrt{x}$ take $(a-b)\sqrt{x}$. *Ans.* $2b\sqrt{x}$.

11. From $a\sqrt{x}+b\sqrt{y}+c\sqrt{z}$ take $d\sqrt{x}+e\sqrt{y}+f\sqrt{z}$.
Ans. $(a-d)\sqrt{x}+(b-e)\sqrt{y}+(c-f)\sqrt{z}$.

12. From $3ax+3ay+3az$ take $3bx+3by+3bz$.
Ans. $3(a-b)(x+y+z)$.

16. Principles relating to the Signs of Aggregation.

$$1.\ a+(b-c)=a+b-c.$$

This formula is true, since the quantity within the parenthesis is to be added to the preceding quantity.

Hence, if a quantity within a sign of aggregation is preceded by the sign +, the sign of aggregation can be omitted without any change of the signs of the terms within the sign of aggregation. Conversely, any number of the terms of a polynomial can be enclosed by a sign of aggregation preceded by the sign +, without any change of the signs of the terms.

$$2.\ a-(b-c)=a-b+c.$$

This formula is true, since the quantity within the parenthesis is to be subtracted from the preceding quantity.

Hence, if a quantity within a sign of aggregation is preceded by the sign —, the sign of aggregation can be omitted, if the sign of every term enclosed by the sign of aggregation be changed. Conversely, any number of terms can be enclosed by a sign of aggregation preceded by the sign —, if the signs of all the terms placed within the sign of aggregation be changed.

The sign preceding the sign of aggregation is the *sign of operation.*

The signs within the sign of aggregation are the *signs of the quantities.*

The signs after the sign of aggregation has been omitted are the *essential signs.*

If the sign of operation is $+$, the essential signs are like the signs of the quantities.

If the sign of operation is $-$, the essential signs are unlike the signs of the quantities.

17. Examples.

Omit the signs of aggregation in the following examples:

1. $a - b + (c - d - e + f)$.
   Ans. $a - b + c - d - e + f$.
2. $a + (-b - c - d + e)$. *Ans.* $a - b - c - d + e$.
3. $a - (-b - c + d - e)$. *Ans.* $a + b + c - d + e$.
4. $a + b - [c - d - (e - f)]$.
   Ans. $a + b - c + d + e - f$.
5. $a - \{b + [d - e - (f - g)]\}$.
   Ans. $a - b - d + e + f - g$.

MULTIPLICATION.

18. Definitions.

1. Multiplication is the process of taking one quantity as many times as there are units in another.

2. The quantities considered are the multiplicand, the multiplier, and the product.

1st. *The multiplicand* is the quantity to be repeated.

2d. *The multiplier* is the quantity which expresses how many times the multiplicand is to be repeated.

3d. *The product* is the amount of the multiplicand taken as many times as there are units in the multiplier.

4th. The multiplicand and multiplier are factors of the product, and the product is the multiple of either factor.

MULTIPLICATION OF MONOMIALS.

19. The Signs.

1. $\begin{cases} +b \times +3 = +b+b+b = +3b. \\ +b \times +a = +b+b+b+\ldots \text{ to } a \text{ terms} = +ab. \end{cases}$

$$\therefore \ (1) \quad + \times + = +.$$

2. $\begin{cases} -b \times +3 = -b-b-b = -3b. \\ -b \times +a = -b-b-b-\ldots \text{ to } a \text{ terms} = -ab. \end{cases}$

$$\therefore \ (2) \quad - \times + = -.$$

3. $\begin{cases} +b \times -3 = -b-b-b = -3b;\ +b \text{ is subtracted 3 times.} \\ +b \times -a = -b-b-b-\ldots \text{ to } a \text{ terms} = -ab. \end{cases}$

$$\therefore \ (3) \quad + \times - = -.$$

4. $\begin{cases} -b \times -3 = +b+b+b = +3b;\ -b \text{ is subtracted 3 times.} \\ -b \times -a = +b+b+b+\ldots \text{ to } a \text{ terms} = +ab. \end{cases}$

$$\therefore \ (4) \quad - \times - = +.$$

From (1) and (4), and (2) and (3), we have the laws:

1*st*. *Like signs give* $+$.
2*d*. *Unlike signs give* $-$.

20. Consequences.

1. *Changing the sign of either factor, changes the sign of the product.*

For,
- 1*st*. If the signs are alike, the product is $+$; changing either sign, the signs will be unlike, and the product $-$.
- 2*d*. If the signs are unlike, the product is $-$; changing either sign, the signs will be alike, and the product $+$.

2. *Changing the signs of both factors, does not change the sign of the product.*

For,
- 1st. If the signs are alike, the product is $+$; changing both signs, the signs will still be alike, and the product $+$.
- 2d. If the signs are unlike, the product is $-$; changing both signs, the signs will still be unlike, and the product $-$.

3. *If the number of minus factors of a product is odd, the product is $-$, otherwise the product is $+$.*

For, the product of the positive factors, if any, is $+$, and this by one minus factor, $-$, by two, $+$, by three, $-$, by four, $+$, in general, by an odd number, $-$, by an even number, $+$

Corollary.—An odd power of a negative quantity is $-$, an even power, $+$.

21. The Co-efficients and Exponents.

1. $5ab^2 \times 4abc = 5abb \times 4abc = 5 \times 4aabbbc = 20a^2b^3c$.

2. $4a^m b^n c^p \times 3a^n b^2 cd^2 = 12a^{m+n} b^{n+2} c^{p+1} d^2$.

Hence, the co-efficients are multiplied, and the exponents of the same letters are added.

22. Rule.

1. *Multiply the co-efficients, prefixing the sign $-$, if the number of minus factors is odd, otherwise the sign $+$.*

2. *Annex all the letters common to two or more of the factors, giving to each an exponent equal to the sum of its exponents in the factors.*

3. *Annex the letters not common with their exponents.*

23. Examples.

1. $5a^3b^2c$ and $7a^2b^4c^2$. *Ans.* $35a^5b^6c^3$.
2. $-7a^7b^2c^8$ and $8ab^2cd$. *Ans.* $-56a^8b^4c^4d$.
3. $10a^2bc$ and $-3ab^3c$. *Ans.* $-30a^3b^4c^2$.
4. $-5x^3y^2$ and $-3x^2y^3z$. *Ans.* $15x^5y^5z$.
5. a^2b, ab^2, ac^2, $-ad$, $-a^2$, $-b^2$. *Ans.* $-a^7b^5c^2d$.
6. $2x^2y$, $3xy^2$, xz^2, $-xz$, $-x^3z$. *Ans.* $6x^8y^3z^4$.
7. a^m, b^n, $-c^p$, $-a^p$, b^p, $-d^q$. *Ans.* $-a^{m+p}b^{n+p}c^pd^q$.
8. $7(a+b)^m$, $-8(a+b)^n$, $-(a+b)$.
   Ans. $56(a+b)^{m+n+1}$.
9. $3(a+b)(c+d)$, $-5(a+b)^3$, $5(c+d)^2$.
   Ans. $-75(a+b)^4(c+d)^3$.
10. $5(a+b)^m(c+d)^n$, $-4(a+b)^3$, $-(c+d)^2$.
    Ans. $20(a+b)^{m+3}(c+d)^{n+2}$.

MULTIPLICATION OF POLYNOMIALS.

24. Illustrations.

1. Multiply $a-b$ by c.

OPERATION.

$$\begin{array}{l} a-b \\ \underline{\quad c \quad} \\ ac-bc \end{array}$$

We first multiply a by c, and obtain ac; but since a is too great by b, the product, ac, is too great by bc. Subtracting bc from ac, we have $ac-bc$ for the true product.

2. Multiply $a-b$ by $c-d$.

OPERATION.

$$\begin{array}{l} a-b \\ \underline{c-d \qquad\qquad} \\ ac-bc-ad+bd \end{array}$$

We first multiply $a-b$ by c, and obtain $ac-bc$; but since c is too great by d, the product is too great by $a-b$ multiplied by d, which is $ad-bd$. Subtracting $ad-bd$ from $ac-bc$, we have $ac-bc-ad+bd$ for the true product.

These examples can be thus performed:

1. $(a - b)\,c = ac - bc.$

2. $(a - b)\,(c - d) = (a - b)\,c - (a - b)\,d$
$$= ac - bc - (ad - bd) = ac - bc - ad + bd.$$

Therefore, for all values of a, b, c, and d, we have the formula,

$$(1)\quad (a - b)\,(c - d) = ac - bc - ad + bd.$$

Hence, for the multiplication of polynomials, as for that of monomials, we have the laws for the signs:

1st. Like signs give $+$.
2d. Unlike signs give $-$.

The laws for the signs in the multiplication of monomials are involved in those for the multiplication of polynomials; for,

1*st*. If $\left\{\begin{matrix} b = 0 \\ d = 0 \end{matrix}\right\}$, (1) becomes (2) $+\,a \times +\,c = +\,ac.$

2*d*. If $\left\{\begin{matrix} a = 0 \\ d = 0 \end{matrix}\right\}$, (1) becomes (3) $-\,b \times +\,c = -\,bc.$

3*d*. If $\left\{\begin{matrix} b = 0 \\ c = 0 \end{matrix}\right\}$, (1) becomes (4) $+\,a \times -\,d = -\,ad.$

4*th*. If $\left\{\begin{matrix} a = 0 \\ c = 0 \end{matrix}\right\}$, (1) becomes (5) $-\,b \times -\,d = +\,bd.$

3. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$.

OPERATION.

$$\begin{array}{lllll}
a^2 & + 2ab & + b^2 & & \\
a^2 & - 2ab & + b^2 & & \\
\hline
a^4 & + 2a^3b & + a^2b^2 & & \\
 & - 2a^3b & - 4a^2b^2 & - 2ab^3 & \\
 & & a^2b^2 & + 2ab^3 & + b^4 \\
\hline
a^4 & & - 2a^2b^2 & & + b^4
\end{array}$$

25. Rule.

Multiply all of the terms of the multiplicand by each term of the multiplier, and add the results.

26. Examples.

1. Multiply $x + y$ by $x + y$. *Ans.* $x^2 + 2xy + y^2$.

2. Multiply $x - y$ by $x - y$. *Ans.* $x^2 - 2xy + y^2$.

3. Multiply $x + y$ by $x - y$. *Ans.* $x^2 - y^2$.

4. Multiply $3x^2 + 2y^2$ by $3x^2 + 2y^2$. *Ans.* $9x^4 + 12x^2y^2 + 4y^4$.

5. Multiply $6x^2y + 5xy^2$ by $6x^2y - 5xy^2$. *Ans.* $36x^4y^2 - 25x^2y^4$.

6. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$. *Ans.* $a^4 + a^2b^2 + b^4$.

7. Multiply $2a^2 - 3ab + 4b^2$ by $5a^2 - 6ab - 2b^2$. *Ans.* $10a^4 - 27a^3b + 34a^2b^2 - 18ab^3 - 8b^4$.

8. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^3 - 3x^2y + 3xy^2 - y^3$. *Ans.* $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.

9. Multiply $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$. *Ans.* $6x^4 - 96$.

10. Multiply $x + y$, $x - y$, $x^2 + xy + y^2$, and $x^2 - xy + y^2$ together. *Ans.* $x^6 - y^6$.

11. Multiply $4(a + b)^2 - 3(c + d)^3$ by $4(a + b)^2 + 3(c + d)^3$. *Ans.* $16(a + b)^4 - 9(c + d)^6$.

12. Multiply $x^m + y^n$ by $x^m - y^n$. *Ans.* $x^{2m} - y^{2n}$.

13. Multiply $x + a$ by $x + b$. *Ans.* $x^2 + (a + b)x + ab$.

14. Multiply $px^m + qx^n + r$ by $px^m + qx^n + r$. *Ans.* $p^2x^{2m} + 2pqx^{m+n} + 2prx^m + q^2x^{2n} + 2qrx^n + r^2$.

15. Prove that $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$, if $ac = b^2$.

MULTIPLICATION BY DETACHED CO-EFFICIENTS.

27. Illustrations.

1. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.

OPERATION.

$$\begin{array}{r}
1 + 3 + 3 + 1 \\
1 + 1 \\
\hline
1 + 3 + 3 + 1 \\
+ 1 + 3 + 3 + 1 \\
\hline
1 + 4 + 6 + 4 + 1
\end{array}$$

The co-efficients are detached from the literal part and multiplied as in multiplication of polynomials. By simple inspection, we see that the exponents of x in the product are 4, 3, 2, 1; those of y, 1, 2, 3, 4. x is not in the last term, nor y in the first term.

$\therefore\ x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 =$ the product.

2. Multiply $x^2 + xy + y^2$ by $x - y$.

OPERATION.

$$\begin{array}{r}
1 + 1 + 1 \\
1 - 1 \\
\hline
1 + 1 + 1 \\
- 1 - 1 - 1 \\
\hline
1 + 0 + 0 - 1
\end{array}$$

The co-efficients are detached and multiplied, and the exponents determined by simple inspection. The terms $0x^2y$ and $0xy^2$ are omitted since they are 0.

$\therefore\ x^3 - y^3 =$ the product.

3. Multiply $x^3 + xy^2 + y^3$ by $x^3 + x^2y + y^3$.

OPERATION.

$$\begin{array}{l}
1 + 0 + 1 + 1 \\
1 + 1 + 0 + 1 \\
\hline
1 + 0 + 1 + 1 \\
\phantom{1 + {}} 1 + 0 + 1 + 1 \\
\phantom{1 + 1 + 1 + {}} 1 + 0 + 1 + 1 \\
\hline
1 + 1 + 1 + 3 + 1 + 1 + 1
\end{array}$$

The terms $0x^2y$ in the multiplicand, and $0xy^2$ in the multiplier, are to be supplied; then the process is as before.

$\therefore\ x^6 + x^5y + x^4y^2 + 3x^3y^3 + x^2y^4 + xy^5 + y^6 =$ the product.

28. Rule.

1. *Arrange the terms of the factors according to the regular descending or ascending powers of a certain letter, supplying missing terms with the co-efficient* 0.

2. *Detach the co-efficients, and multiply as in the multiplication of polynomials.*

3. *Restore the letters with their exponents determined by the law of the product.*

29. Examples.

1. Multiply $x^2 + xy + y^2$ by $x + y$.
Ans. $x^3 + 2x^2y + 2xy^2 + y^3$.

2. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.
Ans. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

3. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
Ans. $a^4 + a^2b^2 + b^4$.

4. Multiply $x^4 + y^4$ by $x^2 + xy + y^2$.
Ans. $x^6 + x^5y + x^4y^2 + x^2y^4 + xy^5 + y^6$.

5. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^3 - 3x^2y + 3xy^2 - y^3$.
Ans. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.

6. Multiply $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
Ans. $x^6 + x^5y + 2x^4y^2 + x^3y^3 + 2x^2y^4 + xy^5 + y^6$.

DIVISION.

30. Definitions.

1. Division is the process of finding one of two factors when their product and one of the factors are known.

2. The quantities considered are the dividend, the divisor, the quotient, and the remainder.

1st. *The dividend* is the given product.

2d. *The divisor* is the given factor of the dividend.

3d. *The quotient* is the required factor of the dividend.

4th. *The remainder* is the dividend minus the product of the divisor and integral part of the quotient. It is an accidental quantity occurring only when the dividend is not a multiple of the divisor, in which case the complete quotient is a mixed quantity.

DIVISION OF MONOMIALS BY MONOMIALS.

31. The Signs.

$$\text{Since, }\left\{\begin{array}{l} +a\times+b=+ab \\ -a\times+b=-ab \\ +a\times-b=-ab \\ -a\times-b=+ab \end{array}\right\}\therefore\left\{\begin{array}{ll} (1) & +ab\div+a=+b \\ (2) & -ab\div-a=+b \\ (3) & -ab\div+a=-b \\ (4) & +ab\div-a=-b \end{array}\right\}$$

From (1) and (2), (3) and (4), we have the laws:

1st. *Like signs give* $+$.
2d. *Unlike signs give* $-$.

32. Consequences.

1. *Changing the sign of either divisor or dividend changes the sign of the quotient.*

For,
- 1st. If the signs are alike, the quotient is $+$; changing either sign, the signs will be unlike, and the quotient $-$.
- 2d. If the signs are unlike, the quotient is $-$; changing either sign, the signs will be alike, and the quotient $+$.

2. *Changing the signs of both divisor and dividend does not change the sign of the quotient.*

For, $\begin{cases} \text{1st. If the signs are alike, the quotient is } +\text{; changing both signs, the signs will still be alike, and the quotient } +. \\ \text{2d. If the signs are unlike, the quotient is } -\text{; changing both signs, the signs will still be unlike, and the quotient } -. \end{cases}$

33. The Co-efficients and Exponents.

1. $10a^m \div 5a^n = 2a^{m-n}$, since $5a^n \times 2a^{m-n} = 10a^m$. Hence,

1st. The co-efficient of the quotient is equal to the co-efficient of the dividend divided by the co-efficient of the divisor.

2d. The exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

2. $\frac{a^n}{a^n} = a^{n-n} = a^0$; but $\frac{a^n}{a^n} = 1$; $\therefore\ a^0 = 1$. Hence,

1st. A quantity with an exponent 0 is equal to 1.

2d. A factor with an exponent 0, can be retained, omitted, or introduced, at pleasure.

3. $\begin{cases} \frac{a^m}{a^n} = \frac{a^{m-n}}{1}, \text{ by dividing both terms by } a^n. \\ \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ by dividing both terms by } a^m. \end{cases}$

$$\therefore \frac{a^{m-n}}{1} = \frac{1}{a^{n-m}}. \quad \text{Hence,}$$

1st. A quantity with any exponent is equal to the reciprocal of the same quantity with the sign of its exponent changed.

2d. A factor may be transferred from one term of a fraction to the other, if the sign of its exponent be changed.

3d. By this transfer, an expression can be freed from negative exponents.

$$\text{Thus,}\ a^2 b^{-3} = \frac{a^2}{b^3},\quad \frac{a^m b^{-n}}{c^p d^{-q}} = \frac{a^m d^q}{c^p b^n}.$$

$$30a^5 b^2 c \div 5a^2 b^2 c^2 d^2 = 6a^3 b^0 c^{-1} d^{-2} = \frac{6a^3}{cd^2}.$$

34. Rule.

1. *Divide the co-efficient of the dividend by the co-efficient of the divisor, prefixing the sign* $+$, *if the signs are alike;* $-$, *if the signs are unlike.*

2. *Annex all the letters common to the dividend and divisor, affecting each with an exponent equal to its exponent in the dividend minus its exponent in the divisor, omitting any letter whose exponent becomes* 0, *unless it is desired to retain a trace of that letter.*

3. *Annex the letters of the dividend not found in the divisor with their exponents.*

4. *Annex the letters of the divisor not found in the dividend with the signs of their exponents changed.*

5. *Free the result from negative exponents, if any.*

35. Examples.

1. Divide $15a^3 b^4 c$ by $5ab^2 c$. *Ans.* $3a^2 b^2$.

2. Divide $36ab^4 c^2$ by $3a^2 b^2 cd$.
Ans. $12a^{-1} b^2 cd^{-1} = \dfrac{12b^2 c}{ad}$.

3. Divide $84a^3 b^3 c$ by $7a^4 b^3 c^2 d^2$. *Ans.* $\dfrac{12}{acd^2}$.

4. Divide $32a^m b^n c^q$ by $8a^n b^n c^r$. *Ans.* $4a^{m-n} c^{q-r}$.

5. Divide $-144a^3b^7c^5$ by $9a^5b^9cd^3$. *Ans.* $-\dfrac{16c^5}{a^2b^2d^3}$.

6. Divide $28a^mb^nc^p$ by $-4a^{-m}b^{-n}c^{-p}$.
Ans. $-7a^{2m}b^{2n}c^{2p}$.

7. Divide $-14a^mb^n$ by $-7a^nb^mc^{-p}$.
Ans. $2a^{m-n}b^{n-m}c^p$.

8. Divide $a^{-m}b^{-n}c^{-p}$ by $72a^mb^nc^p$. *Ans.* $\dfrac{1}{72a^{2m}b^{2n}c^{2p}}$.

9. Divide $30(a+b)^2(x+y)$ by $6(a+b)(x+y)^2$.
Ans. $\dfrac{5(a+b)}{x+y}$.

10. Divide $150(a-b)^5(x-y)^4$ by $10(a-b)(x+y)^2$.
Ans. $\dfrac{15(a+b)^4(x-y)^4}{(x+y)^2}$.

36. General Principles.

$$(1)\qquad mnpq \div pq = mn.$$

$$(2)\qquad \left\{\begin{array}{rcl} mnp^2q \div pq &=& mnp. \\ mnpq \div q &=& mnp. \end{array}\right\}$$

$$(3)\qquad \left\{\begin{array}{rcl} npq \div pq &=& n. \\ mnpq \div mpq &=& n. \end{array}\right\}$$

$$(4)\qquad \left\{\begin{array}{rcl} m^2npq \div mpq &=& mn. \\ mnq \div q &=& mn. \end{array}\right\}$$

Comparing (1) with (2), (3), and (4), we find that,

1st. Multiplying the dividend or dividing the divisor by any quantity multiplies the quotient by that quantity.

2d. Dividing the dividend or multiplying the divisor by any quantity divides the quotient by that quantity.

3d. Multiplying or dividing both dividend and divisor by the same quantity does not ehange the quotient.

DIVISION OF POLYNOMIALS BY MONOMIALS.

37. Illustration.

Divide $6a^3b^2 - 12a^3b^3 + 6a^2b^3$ by $6a^2b^2$.

OPERATION.

$$6a^2b^2)\underline{\,6a^3b^2 - 12a^3b^3 + 6a^2b^3\,}$$
$$a - 2ab + b$$

38. Rule.

Divide each term of the dividend by the divisor, connecting the terms of the quotient with their proper signs.

39. Examples.

1. Divide $3a^3b + 6a^2b^2 + 3ab^3$ by $3ab$.
Ans. $a^2 + 2ab + b^2$.

2. Divide $4x^3y^2 + 8x^3y^3 + 4x^2y^3$ by $4xy$.
Ans. $x^2y + 2x^2y^2 + xy^2$.

3. Divide $15a^3bc - 25ab^3c$ by $5abc$. *Ans.* $3a^2 - 5b^2$.

4. Divide $10a^mb^n - 10a^nb^m$ by $2a^pb^p$.
Ans. $5a^{m-p}b^{n-p} - 5a^{n-p}b^{m-p}$.

5. Divide $2a^3b^2 - 8a^4b^4 + 2a^2b^3$ by $-2a^3b^3$.
Ans. $-a^0b^{-1} + 4ab - a^{-1}b^0 = -\frac{1}{b} + 4ab - \frac{1}{a}$.

DIVISION OF MONOMIALS BY POLYNOMIALS.

40. Illustration.

Divide a by $a + b$.

FIRST OPERATION.

$$a \div (a + b) = \frac{a}{a + b}.$$

SECOND OPERATION.

$$
\begin{array}{l|l}
a & a+b \\
\cline{2-2}
\underline{a+b} & 1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\dots \\
\quad -b & \\
\underline{\quad -b-\frac{b^2}{a}} & \\
\qquad \frac{b^2}{a} & \\
\qquad \underline{\frac{b^2}{a}+\frac{b^3}{a^2}} & \\
\qquad\qquad -\frac{b^3}{a^2} &
\end{array}
$$

Dividing the dividend by the first term of the divisor, we have the quotient 1. Multiplying the divisor by this term of the quotient, and subtracting the product from the dividend, we have the remainder $-b$. Dividing this remainder by the first term of the divisor, and continuing the process, we find the quotient to be an infinite series whose law is, any term is found by multiplying the preceding term by $-\frac{b}{a}$.

41. Rule 1.

Write the quotient as a fraction having the dividend for the numerator and the divisor for the denominator.

42. Rule 2.

1. *Divide the dividend by the first term of the divisor for the first term of the quotient.*

2. *Multiply the divisor by the first term of the quotient and subtract the product from the dividend.*

3. *Divide the first term of the remainder by the first term of the divisor, and continue the process, at pleasure, or till the law of the quotient is obtained.*

43. Examples.

1. Divide a by $a-b$.

Ans. $\frac{a}{a-b}=1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}+\dots$

2. Divide a by $a+b+c$.

$$Ans.\ \frac{a}{a+b+c} = 1 - \frac{b}{a} - \frac{c}{a} + \frac{b^2}{a^2} + \frac{c^2}{a^2} + \ldots$$

3. Divide 1 by $x-y$.

$$Ans.\ \frac{1}{x-y} = \frac{1}{x} + \frac{y}{x^2} + \frac{y^2}{x^3} + \ldots$$

4. Divide 1 by $x+y$.

$$Ans.\ \frac{1}{x+y} = \frac{1}{x} - \frac{y}{x^2} + \frac{y^2}{x^3} - \ldots$$

5. Divide a by $1-a$.

$$Ans.\ \frac{a}{1-a} = a + a^2 + a^3 + \ldots$$

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

44. Illustration.

Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

OPERATION.

$$\begin{array}{rrrr|l}
a^3 & -3a^2b & +3ab^2 & -b^3 & a-b \\
\cline{5-5}
a^3 & -a^2b & & & a^2-2ab+b^2 \\
\cline{1-2}
 & -2a^2b & +3ab^2 & & \\
 & -2a^2b & +2ab^2 & & \\
\cline{2-3}
 & & ab^2 & -b^3 & \\
 & & ab^2 & -b^3 & \\
\cline{3-4}
\end{array}$$

Arranging the polynomials with reference to the descending powers of a, and dividing the first term of the dividend by the first term of the divisor, we find a^2 for the first term of the quotient. Multiplying the divisor by this term of the quotient, and subtracting the product from the dividend, and proceeding as before, we find the entire quotient to be $a^2 - 2ab + b^2$.

45. Rule.

1. *Arrange the dividend and divisor with reference to the ascending or descending powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.*

3. *Multiply the divisor by this term of the quotient and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor, and continue the operation till the remainder is* 0, *or till its first term is not divisible by the first term of the divisor; in which case, write the remainder over the divisor for the fractional part of the quotient, which annex to the integral part of the quotient, for the complete quotient, or continue the process of division in an infinite series.*

46. Examples.

1. Divide $x^2 + 2xy + y^2$ by $x + y$. *Ans.* $x + y$.
2. Divide $x^2 - 2xy + y^2$ by $x - y$. *Ans.* $x - y$.
3. Divide $x^2 - y^2$ by $x + y$. *Ans.* $x - y$.
4. Divide $x^2 - y^2$ by $x - y$. *Ans.* $x + y$.
5. Divide $x^3 - y^3$ by $x - y$. *Ans.* $x^2 + xy + y^2$.
6. Divide $x^4 - y^4$ by $x - y$.
   Ans. $x^3 + x^2y + xy^2 + y^3$.
7. Divide $x^4 - y^4$ by $x + y$.
   Ans. $x^3 - x^2y + xy^2 - y^3$.
8. Divide $x^5 - y^5$ by $x - y$.
   Ans. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
9. Divide $x^5 + y^5$ by $x + y$.
   Ans. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
10. Divide $x^3 - 8$ by $x - 2$. *Ans.* $x^2 + 2x + 4$.

11. Divide $36x^4y^2 - 25x^2y^4$ by $6x^2y - 5xy^2$.
Ans. $6x^2y + 5xy^2$.

12. Divide $49x^4y^6 - 64x^6y^8$ by $7x^2y^3 + 8x^3y^4$.
Ans. $7x^2y^3 - 8x^3y^4$.

13. Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.
Ans. $a^2 - ab + b^2$.

14. Divide $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.
Ans. $a^2 + ab + b^2$.

15. Divide $12x^4 - 192$ by $6x - 12$.
Ans. $2x^3 + 4x^2 + 8x + 16$.

16. Divide $10a^4 - 27a^3b + 34a^2b^2 - 18ab^3 - 8b^4$ by $5a^2 - 6ab - 2b^2$.
Ans. $2a^2 - 3ab + 4b^2$.

17. Divide $x^8 + x^4y^4 + y^8$ by $x^4 + x^2y^2 + y^4$.
Ans. $x^4 - x^2y^2 + y^4$.

18. Divide $x^6 - y^6$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^3y - xy^3 - y^4$.

19. Divide $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$ by $-5a^2 + 4bd - 8cf$.
Ans. $3a^2 - 5bd + cf$.

20. Divide $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 + 3x^2y + 3xy^2 + y^3$.
Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

21. Divide $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.
Ans. $a^3 + 3a^2b + 3ab^2 + b^3$.

22. Divide $x^{2m} - y^{2n}$ by $x^m - y^n$. *Ans.* $x^m + y^n$.

23. Divide $x^2 + (a + b)x + ab$ by $x + a$. *Ans.* $x + b$.

24. Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $x + a$.
Ans. $x^2 + (b + c)x + bc$.

25. Divide $ad + bd + cd + ae + be + ce + af + bf + cf$ by $a + b + c$.
Ans. $d + e + f$.

26. Divide $x^4 + y^4$ by $x + y$.
Ans. $x^3 - x^2y + xy^2 - y^3 + \dfrac{2y^4}{x + y}$.

27. Divide $a^2 - b^2$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

Ans. $a^{\frac{3}{2}} + ab^{\frac{1}{2}} + a^{\frac{1}{2}}b + b^{\frac{3}{2}}$.

28. Divide $a^3 - b^2$ by $a^{\frac{3}{5}} - b^{\frac{2}{5}}$.

Ans. $a^{\frac{12}{5}} + a^{\frac{9}{5}}b^{\frac{2}{5}} + a^{\frac{6}{5}}b^{\frac{4}{5}} + a^{\frac{3}{5}}b^{\frac{6}{5}} + b^{\frac{8}{5}}$.

29. Divide $p^2x^{2m} + 2pqx^{m+n} + 2prx^m + q^2x^{2n} + 2qrx^n + r^2$ by $px^m + qx^n + r$. *Ans.* $px^m + qx^n + r$.

30. Divide $x^n - y^n$ by $x - y$.

Ans. $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + x^{n-5}y^4 + \ldots$

31. Divide $x^m - y^n$ by $x^{\frac{m}{r}} - y^{\frac{n}{r}}$.

Ans. $x^{m-\frac{m}{r}} + x^{m-\frac{2m}{r}}y^{\frac{n}{r}} + x^{m-\frac{3m}{r}}y^{\frac{2n}{r}} + x^{m-\frac{4m}{r}}y^{\frac{3n}{r}} + \ldots$

DIVISION BY DETACHED CO-EFFICIENTS.

47. Illustration.

Divide $a^4 + 2a^2b^2 + b^4$ by $a^2 + b^2$.

OPERATION.

$$
\begin{array}{r|l}
1+0+2+0+1 & 1+0+1 \\ \cline{2-2}
\underline{1+0+1} & 1+0+1 \\
0+1+0 & \\
\underline{0+0+0} & \\
1+0+1 & \\
\underline{1+0+1} &
\end{array}
$$

Supplying the missing terms, we have $a^4 + 0a^3b + 2a^2b^2 + 0ab^3 + b^2$ to be divided by $a^2 + 0ab + b^2$. Detaching the co-efficients and dividing as in division of polynomials, we find the co-efficients of the quotient to be 1, 0, 1; the exponents are determined by inspection. Then restoring the literal part, omitting the term whose co-efficient is 0, we have the quotient $a^2 + b^2$.

48. Rule.

1. *Arrange the dividend and divisor according to the powers of the same letter, supplying missing terms with the co-efficient,* 0.

2. *Detach the co-efficients and divide as in division of polynomials.*

3. *Restore the letters with their exponents determined by the law of the quotient.*

49. Examples.

1. Divide $x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$ by $x + y$.
Ans. $x^3 + 3x^2 y + 3xy^2 + y^3$.

2. Divide $x^3 - y^3$ by $x - y$. *Ans.* $x^2 + xy + y^2$.

3. Divide $x^6 + x^5 y + x^4 y^2 + 3x^3 y^3 + x^2 y^4 + xy^5 + y^6$ by $x^3 + x^2 y + y^3$. *Ans.* $x^3 + xy^2 + y^3$.

4. Divide $x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$. *Ans.* $x^3 + 3x^2 y + 3xy^2 + y^3$.

5. Divide $a^4 + a^2 b^2 + b^4$ by $a^2 + ab + b^2$.
Ans. $a^2 - ab + b^2$.

6. Divide $x^6 + x^5 y + x^4 y^2 + x^2 y^4 + xy^5 + y^6$ by $x^4 + y^4$. *Ans.* $x^2 + xy + y^2$.

7. Divide $x^6 - 3x^4 y^2 + 3x^2 y^4 - y^6$ by $x^3 - 3x^2 y + 3xy^2 - y^3$. *Ans.* $x^3 + 3x^2 y + 3xy^2 + y^3$.

8. Divide $x^6 + x^5 y + 2x^4 y^2 + x^3 y^3 + 2x^2 y^4 + xy^5 + y^6$ by $x^4 + x^2 y^2 + y^4$. *Ans.* $x^2 + xy + y^2$.

9. Divide $x^8 - y^8$ by $x - y$.
Ans. $x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7$.

10. Multiply $x^6 - y^6$ by $x^4 - y^4$, and divide the product by $x^2 + y^2$. *Ans.* $x^8 - x^6y^2 - x^2y^6 + y^8$.

THEOREMS AND FACTORING.

50. Theorem I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \qquad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

Let a denote the first quantity and b the second; then $a + b$ will denote their sum. Now, the square of the sum is the product obtained by multiplying the sum by itself. By actual multiplication, we find that the product is $a^2 + 2ab + b^2$ of which a^2 is the square of the first, $2ab$, twice the product of the first by the second, and b^2, the square of the second. Hence, the theorem is proved, and we have the formula,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Conversely, the sum of the squares of two quantities, plus twice their product is equal to the square of their sum.

Reversing the formula, and writing $(a + b)\ (a + b)$ for $(a + b)^2$, we have the formula for factoring:

$$a^2 + 2ab + b^2 = (a + b)\ (a + b).$$

51. Examples.

1. $3x\ (a + b)^2 = 3x\ (a^2 + 2ab + b^2) = 3a^2x + 6abx + 3b^2x.$

2. $4m^2pq + 8mnpq + 4n^2pq = 4pq\ (m^2 + 2mn + n^2) = 4pq\ (m + n)\ (m + n).$

3. Develop $(x + y)^2$.

4. Square $m + n$.

5. Square $3a^2b + 4a^2b^3$.

6. Develop $4ab\,(c^2 + d^2)^2$

7. Factor $x^2 + 2xy + y^2$.

8. Factor $m^2 + n^2 + 2mn$.

9. Factor $9a^4b^2 + 24a^4b^4 + 16a^4b^6$.

10. Develop $[(a+b)+(c+d)][(a+b)+(c+d)]$.

11. Factor $a^2 + 2ab + b^2 + 2ac + 2ad + 2bc + 2bd + c^2 + 2cd + d^2$.

52. Theorem II.

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

Let the student demonstrate this theorem, and obtain the formula,

$$(a-b)^2 = a^2 - 2ab + b^2.$$

COROLLARY 1.—*The sum of the squares of two unequal quantities is greater than twice their product.*

Since the square of any quantity, positive or negative, is positive, $(a-b)^2$ is positive; $\therefore$ its equal, $a^2 - 2ab + b^2$, is positive.

$$\therefore\ a^2 + b^2 > 2ab, \text{ if } a > b \text{ or } a < b.$$

COROLLARY 2.—*The sum of the squares of two equal quantities is equal to twice their product.*

If $a = b$, $a - b = 0$, $\therefore (a-b)^2 = 0$, $\therefore a^2 - 2ab + b^2 = 0$.

$$\therefore\ a^2 + b^2 = 2ab, \text{ if } a = b$$

Conversely, the sum of the squares of two quantities, minus twice their product, is equal to the square of their difference.

Reversing the formula, we have

$$a^2 - 2ab + b^2 = (a-b)\,(a-b).$$

53. Examples.

1. Develop $(x-y)^2$.
2. Square $2a-3b$.
3. Square $5a^2-4ab$.
4. Develop $5a(a-b)^2$.
5. Develop $6xy(2x-3y)^2$.
6. Factor $x^2-2xy+y^2$.
7. Factor $4a^2-12ab+9b^2$.
8. Factor $25a^4-40a^3b+16a^2b^2$.
9. Factor $5a^3-10a^2b+5ab^2$.
10. Factor $24x^3y-72x^2y^2+54xy^3$.
11. Develop $[(a-b)-(c-d)]^2$.
12. Factor $a^2-2ab+b^2-2ac+2bc+2ad-2bd+c^2+d^2-2cd$.

54. Theorem III.

The product of the sum and difference of two quantities is equal to the difference of their squares.

By actual multiplication, we find

$$(a+b)(a-b)=a^2-b^2.$$

Conversely, the difference of the squares of two quantities is equal to the product of their sum and difference.

By reversing the above formula, we have

$$a^2-b^2=(a+b)(a-b).$$

55. Examples.

1. Develop $(x+y)(x-y)$.
2. Develop $(2a+3b)(2a-3b)$.

3. Develop $(m^2+n^2)(m^2-n^2)$.
4. Develop $(p^2+q^2)(p+q)(p-q)$.
5. Develop $x(8x^3+27y^3)(8x^3-27y^3)$.
6. Factor x^2-y^2.
7. Factor $4a^2-9b^2$.
8. Factor $a^8-a^2b^6$.
9. Factor $50a^2b^2cx-18a^2cx$.
10. Factor $3m^4x-3n^4x$.

56. Theorem IV.

The difference of the same powers of two quantities is divisible by the difference of the quantities.

Let a^m and b^n be two quantities, then will a^{mp} and b^{np} be their p^{th} powers; for,

$$(a^m)^p = a^m \times a^m \times a^m \times \ldots = a^{m+m+m+\ldots\text{ to } p \text{ terms}} = a^{mp},$$
$$\text{and } (b^n)^p = b^n \times b^n \times b^n \times \ldots = b^{n+n+n+\ldots\text{ to } p \text{ terms}} = b^{np}.$$

We are to prove that $a^{mp}-b^{np}$ is divisible by a^m-b^n. This will be accomplished if we prove that the final remainder is 0.

$$\begin{array}{l|l} a^{mp}-b^{np} & a^m-b^n \\ \hline a^{mp}-a^{mp-m}b^n & a^{mp-m}+a^{mp-2m}b^n+a^{mp-3m}b^{2n}+\ldots \end{array}$$

$$\text{1st rem.} = a^{mp-m}b^n - b^{np}$$
$$\underline{a^{mp-m}b^n - a^{mp-2m}b^{2n}}$$
$$\text{2d remainder} = a^{mp-2m}b^{2n} - b^{np}$$
$$\underline{a^{mp-2m}b^{2n} - a^{mp-3m}b^{3n}}$$
$$\text{3d remainder} = a^{mp-3m}b^{3n} - b^{np}$$
$$\ldots\ldots\ldots\ldots$$

$$p^{th} \text{ remainder} = a^{mp-pm}b^{pn} - b^{np} = a^0b^{pn} - b^{np}$$
$$= b^{pn} - b^{np} = 0.$$

$$\therefore (1)\ \frac{a^{mp}-b^{np}}{a^m-b^n} = a^{mp-m} + a^{mp-2m}b^n + \ldots + b^{np-n}.$$

57. The Laws of the Quotient.

1. The signs of the terms are all plus.

2. The co-efficients of the terms are all unity.

3. The exponent of the leading letter in the first term is equal to its exponent in the dividend, minus its exponent in the divisor; and in each succeeding term its exponent is equal to its exponent in the preceding term, minus its exponent in the divisor, and in the last term its exponent is 0, or that letter disappears.

4. The exponent of the other letter in the first term is 0; in the second term its exponent is the same as in the divisor; and in each succeeding term its exponent is equal to its exponent in the preceding, plus its exponent in the divisor; and in the last term its exponent is equal to its exponent in the dividend, minus its exponent in the divisor.

Since the dividend is equal to the product of the divisor and quotient, formula (1) gives, for factoring, formula

$$(2)\quad a^{mp} - b^{np} = (a^m - b^n)\,(a^{mp-m} + a^{mp-2m}b^n + \ldots).$$

58. Examples.

1. $(a^5 - b^6) \div (a - b)$.
2. $(x^7 - y^7) \div (x - y)$.
3. $(a^9 - b^6) \div (a^3 - b^2)$
4. $(x^{12} - y^{18}) \div (x^2 - y^3)$.
5. $(a^{\frac{7}{2}} - b^{\frac{5}{2}}) \div (a^{\frac{7}{6}} - b^{\frac{5}{6}})$.
6. $(a^{12} - b^9) \div (a^4 - b^3)$.
7. $(a^{21} - b^{14}) \div (a^3 - b^2)$.
8. Factor $a^{24} - b^{18}$.
9. Factor $a^n - b^n$.
10. Factor $x^{9m} - y^{6n}$.
11. Factor $m^6 - n^6$.
12. Factor $a^8 - b^6$.
13. Factor $x^{16} - y^{24}$.
14. Factor $x^{49} - y^{35}$.

59. Theorem V.

The difference of the same even powers of two quantities is divisible by the sum of the quantities.

We are to prove that $a^{mp} - b^{np}$ is divisible by $a^m + b^n$, if p is even.

$$
\begin{array}{rl|l}
 & a^{mp} - b^{np} & a^m + b^n \\
\cline{3-3}
 & \underline{a^{mp} + a^{mp-m} b^n} & a^{mp-m} - a^{mp-2m} b^n + a^{mp-3m} b^{2n} - \ldots \\
\text{1st rem.} = & - a^{mp-m} b^n - b^{np} & \\
 & \underline{- a^{mp-m} b^n - a^{mp-2m} b^{2n}} & \\
\text{2d remainder} = & a^{mp-2m} b^{2n} - b^{np} & \\
 & \underline{a^{mp-2m} b^{2n} + a^{mp-3m} b^{3n}} & \\
\text{3d remainder} = & - a^{mp-3m} b^{3n} - b^{np} & \\
 & \ldots\ldots\ldots\ldots\ldots &
\end{array}
$$

We see that the first term of each remainder of an even order is positive.

$\therefore$ If p is even, the p^{th} remainder $= a^{mp-pm} b^{pn} - b^{np} = 0$.

$$\therefore \ (1) \ \frac{a^{mp} - b^{np}}{a^m + b^n} = a^{mp-m} - a^{mp-2m} b^n + \ldots - b^{np-n}.$$

$$\therefore \ (2) \ a^{mp} - b^{np} = (a^m + b^n)(a^{mp-m} - a^{mp-2m} b^n + \ldots).$$

Scholium.—Let the student state the laws of the quotient.

60. Examples.

1. $(a^4 - b^4) \div (a + b)$.
2. $(x^{12} - y^6) \div (a^2 + b)$.
3. $(a^{24} - b^{16}) \div (a^3 + b^2)$.
4. $(x^{\frac{7}{2}} - y^{\frac{9}{2}}) \div (x^{\frac{7}{6}} + y^{\frac{9}{6}})$.
5. $(a^{12} - b^{18}) \div (a^2 + b^3)$.
6. $(a^{32} - b^{40}) \div (a^4 + b^5)$.
7. $(x^{100} - y^{80}) \div (x^{10} - y^8)$.
8. Factor $a^6 - b^4$.
9. Factor $x^{12} - y^{10}$.
10. Factor $x^{-10} - y^{-8}$.
11. Factor $a^{6m} - b^{6n}$.
12. Factor $x^{24} - y^{20}$.
13. Factor $x^{2n} - y^{4n}$.
14. Factor $x^{10} - y^8$.

61. Theorem VI.

The sum of the same odd powers of two quantities is divisible by the sum of the quantities.

We are to prove that $a^{mp} + b^{np}$ is divisible by $a^m + b^n$, if p is odd.

$$
\begin{array}{l|l}
a^{mp} + b^{np} & a^m + b^n \\
\underline{a^{mp} + a^{mp-m} b^n} & a^{mp-m} - a^{mp-2m} b^n + a^{mp-3m} b^{2n} - \ldots
\end{array}
$$

$$
\begin{array}{l}
\text{1st rem.} = - a^{mp-m} b^n + b^{np} \\
\qquad\quad \underline{- a^{mp-m} b^n - a^{mp-2m} b^{2n}} \\
\text{2d remainder} = a^{mp-2m} b^{2n} + b^{np} \\
\qquad\qquad\qquad \underline{a^{mp-2m} b^{2n} + a^{mp-3m} b^{3n}} \\
\text{3d remainder} = - a^{mp-3m} b^{3n} + b^{np} \\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots
\end{array}
$$

We see that the first term of each remainder of an odd order is negative.

$\therefore$ If p is odd, the p^{th} remainder $= - a^{mp-pm} b^{pn} + b^{np} = 0$.

$$\therefore \ (1) \quad \frac{a^{mp} + b^{np}}{a^m + b^n} = a^{mp-m} - a^{mp-2m} b^n + \ldots + b^{np-n}.$$

$$\therefore \ (2) \quad a^{mp} + b^{np} = (a^m + b^n)(a^{mp-m} - a^{mp-2m} b^n + \ldots).$$

Scholium.—Let the student state the laws of the quotient.

62. Examples.

1. $(a^9 + b^6) \div (a^3 + b^2)$.
2. $(a^{12} + b^9) \div (a^4 + b^3)$.
3. $(a^7 + b^5) \div (a^{\frac{7}{3}} + b^{\frac{5}{3}})$.
4. $(a^{21} + b^{14}) \div (a^3 + b^2)$.
5. $(x^{3mp} + y^{6np}) \div (x^m + y^{2n})$.
6. $(x^{10} + y^{10}) \div (x^{\frac{10}{3}} + y^{\frac{10}{3}})$.
7. $(x^{25} + y^{20}) \div (x^5 + y^4)$.
8. Factor $a^7 + b^7$.
9. Factor $x^{15} + y^{12}$.
10. Factor $x^{10} + y^5$.
11. Factor $a^9 + b^9$.
12. Factor $a^{5p} + b^{5q}$.
13. Factor $x^{28} + y^{21}$.
14. Factor $x^{13} + y^{26}$.

63. Theorem VII.

The product of two binomials whose first terms are equal is equal to the square of the first, plus the sum of the second terms into the first, plus the product of the second terms.

$$(1)\quad (x + a)(x - b) = x^2 + (a - b)x - ab.$$

Remark.—The sum of a and $-b$ is $a - b$, and plus this sum into x is $+(a - b)x$. The product of a and $-b$ is $-ab$, and plus this product is $+(-ab)$ or $-ab$. The second member of formula (1) is of course found from the first by actual multiplication.

Conversely, a trinomial whose first term is a square, and the square root of whose first term is a factor of the second term, can be resolved into two binomial factors of which the first term in each is the square root of the first term of the trinomial, and the second terms the two quantities whose sum is the co-efficient of the second term of the trinomial, and whose product is the third term.

Reversing formula (1), we have

$$(2)\quad x^2 + (a - b)x - ab = (x + a)(x - b).$$

64. Examples.

1. Develop $(x + 5)(x + 3)$.
2. Develop $(x - 7)(x - 8)$.
3. Develop $(x - 9)(x + 5)$.
4. Develop $(x + 11)(x + 5)$.
5. Develop $(x - 7)(x + 9)$.
6. Develop $(x^2 - 5)(x^2 - 8)$.
7. Develop $(x^m - 5)(x^m + 6)$.

8. Factor $x^2 + 2x - 15$.

9. Factor $x^2 - 4x - 45$.

10. Factor $x^2 - 15x + 56$.

11. Factor $x^2 + 13x + 42$.

12. Factor $x^4 - 13x^2 + 40$.

13. Factor $x^6 - 18x^3 + 77$.

14. Factor $x^8 - 15x^4 + 54$.

65. Miscellaneous Examples.

Factor or develop the following:

1. $(a^2 - b^2)^4$.
2. $[(a + b)(a - b) + c][(a + b)(a - b) - c]$.
3. $(a + b + c)^2 - d^2$.
4. $(a + b + c)(a + b - c)$.
5. $(a + b)^2 - (a - b)^2$.
6. $5a(a - b)^2$.
7. $11a^5b - 176ab^5$.
8. $11a(a^2 + 4b^2)(a + 2b)(a - 2b)$.
9. $x^{15} + x^9$.
10. $a^5 + 32b^5$.
11. $(a^6 - 1)(a^6 + 1)(a^{12} + 1)$.
12. $256a^8 - 6561b^8$.
13. $a^{16} - b^{16}$.
14. $a^4 + a^2b^2 + b^4$.
15. $x^3 + x^2y + xy^2 + y^8$.
16. $5ax^2 + 65ax + 210a$.
17. $(a + b)^3 - (c + d)^3$.
18. $x^5 - xy^4 - x^4y + y^5$.

THE GREATEST COMMON DIVISOR.

66. Definitions.

1. **A divisor** of a quantity is a quantity which will divide the given quantity without a remainder.

2. **A common divisor** of two or more quantities is a divisor of each of those quantities.

3. **The greatest common divisor** of two or more quantities is the greatest quantity which is a divisor of each of those quantities.

4. **A multiple** of a quantity is a quantity of which the given quantity is a factor.

5. **A composite quantity** is a quantity which can be resolved into integral factors differing from the quantity itself and unity.

6. **A prime quantity** is a quantity which can not be resolved into integral factors differing from the quantity itself and unity.

7. Two or more quantities are *relatively prime* when they have no other common integral factor than unity.

67. Notation.

Let c. d. denote a *common divisor*, and g. c. d. denote the *greatest common divisor*.

FIRST METHOD—BY FACTORING.

68. Principles.

1. Any common factor of two or more quantities is a c. d. of those quantities, since a factor of a quantity is a divisor of that quantity.

2. The product of two or more common prime factors is a c. d., since this product is a common factor.

3. The product of all the common prime factors is the g. c. d., since this product is the greatest common factor.

69. Illustrations.

1. Find the g. c. d. of $30a^3b^4c^2$ and $105a^4b^2d$.

OPERATION.

$$\left.\begin{array}{l} 30a^3b^4c^2 = 2 \times 3 \times 5a^3b^4c^2 \\ 105a^4b^2d \ = 3 \times 5 \times 7a^4b^2d \end{array}\right\} \therefore \left\{\begin{array}{l} 3 \times 5a^3b^2 = 15a^3b^2 \\ = \text{g. c. d.} \end{array}\right.$$

2. Find the g. c. d. of $2a^2 - 2b^2$, $2am + 2bm$, and $2a^2 + 4ab + 2b^2$.

OPERATION.

$$\left.\begin{array}{l} 2a^2 - 2b^2 = 2\,(a + b)\,(a - b) \\ 2am + 2bm = 2m\,(a + b) \\ 2a^2 + 4ab + 2b^2 = 2\,(a + b)\,(a + b) \end{array}\right\} \therefore\ 2(a + b) = \text{g. c. d.}$$

70. Rule.

1. *Resolve the quantities into their prime factors.*
2. *Take the product of all the common prime factors.*

71. Examples.

1. Find the g. c. d. of $210a^3b^4c$ and $1155a^4b^3d^2$.
Ans. $105a^3b^3$.

2. Find the g. c. d. of $231a^2b^3c$, $1001a^3bc^2d$, and $154a^2b^2c$.
Ans. $77a^2bc$.

3. Find the g. c. d. of $a^2 - 2ab + b^2$ and $a^2 - b^2$.
Ans. $a - b$.

4. Find the g. c. d. of $a^4 - b^4$ and $a^2 + 2ab + b^2$.
Ans. $a + b$.

5. Find the g. c. d. of $x^2 + 8x + 15$, $x^2 - 2x - 15$, and $x^2 + 2x - 3$.
Ans. $x + 3$.

6. Find the g. c. d. of $x^2 + 8x + 15$, $x^2 + 2x - 15$, $x^2 + 4x - 5$, and $x^2 - 2x - 35$.
Ans. $x + 5$.

7. Find the g. c. d. of $3x(a + b)^2$ and $2ax(a^2 - b^2)$.
Ans. $ax + bx$.

8. Find the g. c. d. of $x^6 - 11x^3 + 30$, $x^6 - 13x^3 + 42$, and $x^6 + x^3 - 42$.
Ans. $x^3 - 6$.

9. Find the g. c. d. of $a^4 + a^2b^2 + b^4$ and $a^3 - 2a^2b + 2ab^2 - b^3$.
Ans. $a^2 - ab + b^2$.

10. Find the g. c. d. of $x^{2m} + x^m - 30$ and $x^{2m} - x^m - 42$.
Ans. $x^m + 6$.

11. Find the g. c. d. of $3ax^6 - 3ay^6$ and $3ax^4y + 3axy^4$.
Ans. $3ax^3 + 3ay^3$.

SECOND METHOD—BY DIVISION.

72. Principles.

1. A divisor of a quantity is a divisor of any multiple of that quantity. Thus, d is a divisor of dq and of mdq.

2. A c. d. of two quantities is a divisor of their sum. Thus, d is a c. d. of dq and dq', and a divisor of $dq + dq'$.

3. A c. d. of two quantities is a divisor of their difference. Thus, d is a c. d. of dq and dq', and a divisor of $dq - dq'$.

4. If the less of two quantities is a divisor of the greater, it is their g. c. d.

5. If two quantities are relatively prime, their g. c. d. is 1.

6. If one of two quantities be either multiplied or divided by a quantity which is prime to the other quantity, the g. c. d. will not be changed. Thus, d is the g. c. d. of ad and bd, also of mad and bd, and of d and bd.

7. The g. c. d. of three quantities is the g. c. d. of the g. c. d. of two of them and the third quantity, and, in general, the g. c. d. of n quantities is the g. c. d. of the g. c. d. of $(n - 1)$ of them and the remaining quantity.

8. Any common factor of two or more quantities is a factor of their g. c. d., and the product of this common factor by the g. c. d. of the quotients obtained by dividing the quantities by this common factor, is the g. c. d. of the quantities. Thus, m is a common factor of $ma^2 - mb^2$ and $ma^2 + 2mab + mb^2$. Dividing by m, the quotients are $a^2 - b^2$ and $a^2 + 2ab + b^2$, of which the g. c. d. is $a + b$, and $m(a + b)$ is the g. c. d. of $ma^2 - mb^2$ and $ma^2 + 2mab + mb^2$.

73. Proposition.

The g. c. d. of the divisor and remainder is the g. c. d. of the divisor and dividend.

Let us take the less of the two quantities whose g. c. d. is to be found for the divisor, and the greater for the dividend, and let d denote the divisor; D, the dividend; q, the quotient; and r, the remainder. Then from the principles of division we have

$$(1)\quad D = dq + r.$$
$$(2)\quad D - dq = r.$$

The demonstration of the above proposition depends on the following

LEMMAS.

1. *Any c. d. of d and r is a c. d. of d and D.*

For, a divisor of d is a divisor of dq, by Principle 1; $\therefore$ a c. d. of d and r is a c. d. of dq and r, and $\therefore$ a divisor of D, by Principle 2, since $D = dq + r$, and $\therefore$ a c. d. of d and D.

2. *Any c. d. of d and D is a c. d. of d and r.*

For, a divisor of d is a divisor of dq, by Principle 1; $\therefore$ a c. d. of d and D is a c. d. of dq and D, and $\therefore$ a divisor of r, by Principle 3, since $D - dq = r$, and $\therefore$ a c. d. of d and r.

It has now been proved,

1. *That any c. d. of d and r is a c. d. of d and D.*
2. *That any c. d. of d and D is a c. d. of d and r.*

Hence, the c. d.'s of d and r are identical with the c. d.'s of d and D; $\therefore$ the g. c. d. in the one case is the g. c. d. in the other case. Hence,

The g. c. d. of d and r is the g. c. d. of d and D.

The process is as follows:

$$\begin{array}{l} d)\ D\ (q \\ \underline{\ dq\ } \\ \quad r)\ d\ (q' \\ \quad \underline{\ rq'\ } \\ \qquad r')\ r\ (q'' \\ \qquad \underline{\ r'q''\ } \\ \qquad\quad r'' \end{array}$$

Let us suppose $r''=0$; then by Principle 4, r' is the g. c. d. of r and r'; but, by the Proposition, the g. c. d. of r and r' is the g. c. d. of r and d; $\therefore$ r' is the g. c. d. of r and d; but, by the Proposition, the g. c. d. of d and r is the g. c. d. of d and D; $\therefore$ r' is the g. c. d. of d and D. Hence, the first remainder, which is a divisor of the last divisor, is the g. c. d. sought.

74. Application.

Find the g. c. d. of the following polynomials:

$9x^5yz - 30x^3yz + 45xyz + 24yz$ and

$15ax^5y - 30ax^4y - 90ax^3y + 60ax^2y + 195axy + 90ay.$

Dividing both polynomials by the common factor $3y$, which is a factor of the g. c. d., the quotients are

$$3x^5z - 10x^3z + 15xz + 8z \text{ and}$$
$$5ax^5 - 10ax^4 - 30ax^3 + 20ax^2 + 65ax + 30a.$$

Dividing the first of these polynomials by z and the second by $5a$, which will not affect the g. c. d. by Art. 72, Principle 6, and taking the first result for the dividend and the second for the divisor, we proceed thus:

$$\begin{array}{l|l} x^5 - 2x^4 - 6x^3 & \\ \quad + 4x^2 + 13x + 6) & 3x^5 - 10x^3 + 15x + 8\ (3 \\ & \underline{3x^5 - 6x^4 - 18x^3 + 12x^2 + 39x + 18} \\ & 6x^4 + 8x^3 - 12x^2 - 24x - 10 \end{array}$$

Since the g. c. d. of d and r is the g. c. d. of d and D, we shall find the g. c. d. of d and r.

Dividing r by 2 and multiplying d by 3, and taking the results for d and D, we proceed thus:

$$
\begin{array}{l|l}
3x^4+4x^3 & \\
-6x^2-12x-5 & 3x^5-6x^4-18x^3+12x^2+39x+18\ (x \\
 & \underline{3x^5+4x^4-6x^3-12x^2-5x} \\
 & 2)\underline{-10x^4-12x^3+24x^2+44x+18} \\
 & -5x^4-6x^3+12x^2+22x+9 \\
 & \underline{\qquad\qquad\qquad\qquad\qquad\qquad 3} \\
 & -15x^4-18x^3+36x^2+66x+27\ (-5 \\
 & \underline{-15x^4-20x^3+30x^2+60x+25} \\
 & \qquad 2x^3+6x^2+6x+2
\end{array}
$$

Dividing r by 2 and taking the result for d, and the last d for D, we proceed thus:

$$
\begin{array}{l|l}
x^3+3x^2+3x+1 & 3x^4+4x^3-6x^2-12x-5\ (3x-5 \\
 & \underline{3x^4+9x^3+9x^2+3x} \\
 & -5x^3-15x^2-15x-5 \\
 & \underline{-5x^3-15x^2-15x-5}
\end{array}
$$

$\therefore\ x^3+3x^2+3x+1$ is the g. c. d. of

$3x^5z-10x^3z+15xz+8z$ and

$5ax^5-10ax^4-30ax^3+20ax^2+65ax+30a.$

$\therefore\ 3y(x^3+3x^2+3x+1)=3x^3y+9x^2y+9xy+3y=$ g. c. d. of the given polynomials.

75. Rule.

1. *Set aside the obvious common factor, if any, as a factor of the g. c. d.*

2. *Divide the quantities whose g. c. d. is to be found by the factor set aside, and, if necessary, prepare the quotients for division by rejecting or introducing a factor in either which is prime to the other.*

3. *Divide one by the other, the divisor by the remainder, and so on, till there is no remainder. The last divisor will be the g. c. d. of the prepared quantities, and this g. c. d. multiplied by the factor set aside will be the g. c. d. of the given quantities.*

4. *If there are more than two quantities, find the g. c. d. of two of them, then of this g. c. d. and the third quantity, and so on.*

76. Examples.

1. Find the g. c. d. of $a^4 - b^4$ and $a^4 + 2a^2b^2 + b^4$. *Ans.* $a^2 + b^2$.

2. Find the g. c. d. of $x^4 + x^2y^2 + y^4$ and $x^3 - y^3$. *Ans.* $x^2 + xy + y^2$.

3. Find the g. c. d. of $a^3 + 3a^2 + 4a + 12$ and $a^3 + 4a^2 + 4a + 3$. *Ans.* $a + 3$.

4. Find the g. c. d. of $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$. *Ans.* $x^2 + x + 1$.

5. Find the g. c. d. of $x^4 - 9x^2 + 20$ and $x^4 + 4x^2 - 32$. *Ans.* $x^2 - 4$.

6. Find the g. c. d. of $x^6 - 15x^3 + 56$, $x^6 + 4x^3 - 96$, and $x^4 - 9x^2 + 20$. *Ans.* $x - 2$.

7. Find the g. c. d. of $4a^2 - 5ab + b^2$, $3a^3 - 3a^2b + ab^2 - b^3$, and $a^4 - b^4$. *Ans.* $a - b$.

8. Find the g. c. d. of $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$. *Ans.* $2x^3 - 4x^2 + x - 1$.

9. Find the g. c. d. of $a^4 - a^3b - a^2b^2 - ab^3 - 2b^4$ and $3a^3 - 7a^2b + 3ab^2 - 2b^3$. *Ans.* $a - 2b$.

10. Find the g. c. d. of $x^5 - x^4y - xy^4 + y^5$, $x^4 - x^3y - x^2y^2 + xy^3$, and $x^3y + x^2y^2 - xy^3 - y^4$. *Ans.* $x^2 - y^2$.

THE LEAST COMMON MULTIPLE.

77. Definitions.

1. **A multiple** of a quantity is a quantity of which the given quantity is a factor.

2. **A common multiple** of two or more quantities is a quantity of which each of the given quantities is a factor.

3. **The least common multiple** of two or more quantities is the least quantity of which each of the given quantities is a factor.

78. Notation.

Let c. m. denote a *common multiple,* and l. c. m. denote the *least common multiple.*

FIRST METHOD—BY FACTORING.

79. Principles.

1. The l. c. m. of two or more quantities contains as factors all the prime factors of these quantities and no other factors.

2. If two quantities are relatively prime, their l. c. m. is their product.

80. Illustration.

Find the l. c. m. of $a^2 - b^2$ and $a^2 - 2ab + b^2$.

OPERATION.

$$\left.\begin{array}{l} a^2 - b^2 = (a+b)(a-b) \\ a^2 - 2ab + b^2 = (a-b)(a-b) \end{array}\right\} \therefore \left\{\begin{array}{l} (a+b)(a-b)(a-b) \\ = \text{l. c. m.} \end{array}\right.$$

81. Rule.

1. *Resolve the quantities into their simplest factors.*

2. *Find the product of the factors, each taken the greatest number of times that it occurs as a factor in any of the given quantities.*

82. Examples.

1. Find the l. c. m. of $a^2 + 2ab + b^2$ and $a^2 - b^2$.
Ans. $a^3 + a^2b - ab^2 - b^3$.

2. Find the l. c. m. of a^2, $a + b$, $a - b$, $a^2 + 2ab + b^2$, $a^2 - b^2$, and $a^2 - 2ab + b^2$. *Ans.* $a^6 - 2a^4b^2 + a^2b^4$.

3. Find the l. c. m. of $x^2 + 5x + 6$, $x^2 + 7x + 10$, and $x^2 + 8x + 15$. *Ans.* $x^3 + 10x^2 + 31x + 30$.

4. Find the l. c. m. of $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, and $(a + b)^3$. *Ans.* $a^5 + a^4b - 2a^3b^2 - 2a^2b^3 + ab^4 + b^5$.

5. Find the l. c. m. of $x - 1$, $x^2 - 1$, $x^3 - 1$, $x^4 - 1$, and $x^5 - 1$.

$$\textit{Ans.} \begin{cases} x^{10} + 2x^9 + 3x^8 + 3x^7 \\ \quad + 2x^6 - 2x^4 - 3x^3 - 3x^2 - 2x - 1. \end{cases}$$

6. Find the l. c. m. of $x^2 + (a + b)x + ab$ and $x^2 + (a - b)x - ab$. *Ans.* $x^3 + ax^2 - b^2x - ab^2$.

SECOND METHOD—BY MEANS OF G. C. D.

83. Proposition.

The l. c. m. of two quantities is equal to their product divided by their g. c. d., or, which is the same in effect, is equal to either quantity multiplied by the quotient which arises from dividing the other by the g. c. d.

Let ad and bd be two quantities whose g. c. d. is d; l. c. m., abd; and product, abd^2. Then,

$$abd = \frac{abd^2}{d} = ad \times \frac{bd}{d} = bd \times \frac{ad}{d}.$$

84. Rule.

1. *If there are two quantities, divide either by their g. c. d. and multiply the other quantity by the quotient.*

2. *If there are more than two quantities, find the l. c. m. of two of them, and the l. c. m. of this l. c. m. and the third, and so on.*

85. Examples.

1. Find the l. c. m. of $a^2 - b^2$ and $a^2 - 2ab + b^2$.
Ans. $a^3 - a^2b - ab^2 + b^3$.

2. Find the l. c. m. of $x^3 - 1$ and $x^2 - 3x + 2$.
Ans. $x^4 - 2x^3 - x + 2$.

3. Find the l. c. m. of $x^2 - 14x + 48$ and $x^2 - 18x + 80$.
Ans. $x^3 - 24x^2 + 188x - 480$.

4. Find the l. c. m. of $x^4 - 2x^2 + 1$ and $x^4 + 4x^3 + 6x^2 + 4x + 1$. *Ans.* $x^6 + 2x^5 - x^4 - 4x^3 - x^2 + 2x + 1$.

5. Find the l. c. m. of $x^4 - 1$, $x^3 + x^2 + x + 1$, and $x^3 - x^2 + x - 1$.
Ans. $x^4 - 1$.

6. Find the l. c. m. of $a^2 - 4b^2$, $a^3 - 2a^2b + 4ab^2 - 8b^3$, and $a^3 + 2a^2b + 4ab^2 + 8b^3$. *Ans.* $a^4 - 16b^4$.

FRACTIONS.

86. Definitions.

1. **A fraction** is one or more of the equal parts of a unit. Thus, if a unit be divided into d equal parts, each part will be $\frac{1}{d}$, which is read 1 divided by d; and n parts will be $\frac{n}{d}$, which is read n divided by d.

2. **The terms** of a fraction are the two quantities by which the fraction is expressed.

1st. *The numerator* is the term above the line; it expresses the number of parts taken.

2d. *The denominator* is the term below the line; it expresses the number of parts into which the unit is divided, the relation of the parts to the unit, the number of parts which equal the unit, and the kind and name of the parts.

3. **An entire quantity** is a quantity which does not contain a fraction.

4. **A mixed quantity** is a quantity which contains both an entire and a fractional part.

5. **A simple fraction** is a fraction whose terms are entire.

6. **A complex fraction** is a fraction having a fraction in either of its terms.

7. **A compound fraction** is a fraction of a fraction.

87. Principles.

1. The numerator of a fraction is the dividend, the denominator is the divisor, and the fraction is the quotient.

2. If the signs of the terms of a fraction are alike, the fraction is plus; if unlike, minus. [Art. 31.]

3. Changing the sign of either term changes the sign of the fraction. [Art. 32, 1.]

4. Changing the signs of both terms does not change the sign of the fraction. [Art. 32, 2].

5. Multiplying the numerator or dividing the denominator by any quantity multiplies the fraction by that quantity. [Art. 36.]

6. Dividing the numerator or multiplying the denominator by any quantity divides the fraction by that quantity.

7. Multiplying or dividing both terms by the same quantity does not change the value of the fraction.

REDUCTION OF FRACTIONS.

88. Definition.

The reduction of a fraction is the process of changing its form without affecting its value.

89. Case I.

To reduce fractions to their lowest terms.

A fraction is in its lowest terms when its numerator and denominator are relatively prime.

The reduction of a fraction to its lowest terms is the process of rejecting the factors common to the numerator and denominator.

90. Rules.

1. *Divide both terms by their g. c. d.*

2. *Divide both terms by any c. d.; in like manner divide both terms of the result, and continue the operation till the terms are relatively prime.*

3. *Resolve the terms into their prime factors, cancel the factors common to both terms, and take the product of the factors remaining in the numerator for a numerator, and the product of the factors remaining in the denominator for the denominator.*

91. Examples.

1. Reduce $\dfrac{a^2-b^2}{a^2+2ab+b^2}=\dfrac{(a+b)\ (a-b)}{(a+b)\ (a+b)}=\dfrac{a-b}{a+b}$.

2. Reduce $\dfrac{a^2-b^2}{a^2-2ab+b^2}$. *Ans.* $\dfrac{a+b}{a-b}$.

3. Reduce $\dfrac{a^2-2ab+b^2}{a^3-b^3}$. *Ans.* $\dfrac{a-b}{a^2+ab+b^2}$.

4. Reduce $\dfrac{x^2+2xy+y^2}{x^3+y^3}$. *Ans.* $\dfrac{x+y}{x^2-xy+y^2}$.

5. Reduce $\dfrac{x^2+13x+42}{x^2+14x+48}$. *Ans.* $\dfrac{x+7}{x+8}$.

6. Reduce $\dfrac{x^4-y^4}{x^6-y^6}$. *Ans.* $\dfrac{x^2+y^2}{x^4+x^2y^2+y^4}$.

7. Reduce $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^3-x^2y-xy^2+y^3}$. *Ans.* $\dfrac{x-y}{x+y}$.

8. Reduce $\dfrac{x^2+(a+b)x+ab}{x^2+(b+c)x+bc}$. *Ans.* $\dfrac{x+a}{x+c}$.

9. Reduce $\dfrac{(a+b)^2+(a-b)^2}{a^4-b^4}$. *Ans.* $\dfrac{2}{a^2-b^2}$.

10. Reduce $\dfrac{x^4-px^3+(q-1)x^2+px-q}{x^4-qx^3+(p-1)x^2+qx-p}$.

Ans. $\dfrac{x^2-px+q}{x^2-qx+p}$.

92. Case II.

To reduce entire or mixed quantities to fractions.

1. $a=\dfrac{a}{1}=\dfrac{ad}{d}$, by multiplying both terms by d.

2. $a+\dfrac{n}{d}=\dfrac{ad}{d}+\dfrac{n}{d}=\dfrac{ad+n}{d}$.

3. $a-\dfrac{b-c}{d}=\dfrac{ad}{d}-\dfrac{b-c}{d}=\dfrac{ad-b+c}{d}$.

93. Rule.

Multiply the entire quantity by the denominator, add or subtract the numerator of the fractional part, if any, according as its sign is + or —, and write the result over the denominator.

94. Examples.

1. Reduce $x + \frac{y^2}{x}$. *Ans.* $\frac{x^2 + y^2}{x}$.

2. Reduce $1 + \frac{a - b}{a + b}$. *Ans.* $\frac{2a}{a + b}$.

3. Reduce $a + b - \frac{(a - b)^2}{a + b}$. *Ans.* $\frac{4ab}{a + b}$.

4. Reduce $x - y - \frac{x^2 - y^2}{x + y}$.

5. Reduce $x + 5 + \frac{x + 3}{x - 7}$. *Ans.* $\frac{x^2 - x - 32}{x - 7}$.

6. Reduce $x + a + \frac{x + b}{x + c}$.

Ans. $\frac{x^2 + (a + c + 1)x + ac + b}{x + c}$.

7. Reduce $(a + b)^2 - \frac{a^4 + b^4}{(a - b)^2}$. *Ans.* $\frac{-2a^2b^2}{a^2 - 2ab + b^2}$.

8. Reduce $x^2 + xy + y^2 + \frac{x^2y^2}{x^2 - xy + y^2}$.

Ans. $\frac{(x^2 + y^2)^2}{x^2 - xy + y^2}$.

9. Reduce $1 + n - \frac{(1 - n)^2}{1 + n}$. *Ans.* $\frac{4n}{1 + n}$.

10. Reduce $a + b - \frac{2b(3a^2 + b^2)}{(a + b)^2}$. *Ans.* $\frac{(a - b)^3}{(a + b)^2}$.

95. Case III.

To reduce a fraction to an entire or mixed quantity.

Since a fraction is the quotient of which the numerator is the dividend and the denominator the divisor, we have the following rule.

96. Rule.

Divide the numerator by the denominator, and write the remainder, if any, over the denominator, for the fractional part, and connect it with the integral part by the sign $+$.

97. Examples.

1. Reduce $\dfrac{x^2+y^2}{x}$. *Ans.* $x+\dfrac{y^2}{x}$.

2. Reduce $\dfrac{x^2-x-32}{x-7}$. *Ans.* $x+6+\dfrac{10}{x-7}$.

3. Reduce $\dfrac{a^4-2a^2b^2+b^4}{a^2+2ab+b^2}$. *Ans.* $a^2-2ab+b^2$.

4. Reduce $\dfrac{x^4-y^4}{x-y}$. *Ans.* $x^3+x^2y+xy^2+y^3$.

5. Reduce $\dfrac{x^5+y^5}{x+y}$. *Ans.* $x^4-x^3y+x^2y^2-xy^3+y^4$.

6. Reduce $\dfrac{a}{a+b}$. *Ans.* $1-\dfrac{b}{a}+\dfrac{b^2}{a^2}-\dfrac{b^3}{a^3+a^2b}$.

7. Reduce $\dfrac{(a+b)^4-(a-b)^4}{8ab}$. *Ans.* a^2+b^2.

8. Reduce $\dfrac{(x^2+y^2)^2}{x^2-xy+y^2}$.

Ans. $x^2+xy+y^2+\dfrac{x^2y^2}{x^2-xy+y^2}$.

9. Reduce $\dfrac{x^2 + (a + c + 1)\,x + ac + b}{x + c}$.

Ans. $x + a + 1 + \dfrac{b - c}{x + c}$.

98. Case IV.

To reduce fractions to equivalent fractions having a common denominator.

The object of this reduction is to prepare the fractions for addition or subtraction.

99. Rule.

Divide the l. c. m. of the denominators by each denominator, and multiply both terms of each fraction by the corresponding quotient.

100. Examples.

Reduce the following fractions to equivalent fractions having a common denominator.

1. $\dfrac{a}{bc}$, $\dfrac{b}{ac}$, and $\dfrac{c}{ab}$.

OPERATION.

$$\frac{a}{bc} = \frac{a \times a}{bc \times a} = \frac{a^2}{abc}.$$

$$\frac{b}{ac} = \frac{b \times b}{ac \times b} = \frac{b^2}{abc}.$$

$$\frac{c}{ab} = \frac{c \times c}{ab \times c} = \frac{c^2}{abc}.$$

The operations are not usually indicated, thus:

$$\frac{a}{bc},\ \frac{b}{ac},\ \frac{c}{ab} \text{ give } \frac{a^2}{abc},\ \frac{b^2}{abc},\ \frac{c^2}{abc}.$$

2. $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$. *Ans.* $\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$.

3. $\frac{a-b}{a+b}, \frac{a+b}{a-b}$. *Ans.* $\frac{a^2-2ab+b^2}{a^2-b^2}, \frac{a^2+2ab+b^2}{a^2-b^2}$.

4. $\frac{ab}{cd}, \frac{cd}{ab}, \frac{ef}{gh}, \frac{gh}{cf}$.

Ans. $\frac{a^2b^2efgh}{abcdefgh}, \frac{c^2d^2efgh}{abcdefgh}, \frac{abcde^2f^2}{abcdefgh}, \frac{abcdg^2h^2}{abcdefgh}$.

5. $\frac{a}{a^2-b^2}, \frac{b}{a+b}, \frac{c}{a-b}$.

Ans. $\frac{a}{a^2-b^2}, \frac{b\,(a-b)}{a^2-b^2}, \frac{c\,(a+b)}{a^2-b^2}$.

6. $\frac{1}{(a+b)^2}, \frac{1}{(a-b)^2}, \frac{1}{(a+b)\,(a-b)}$.

Ans. $\frac{(a-b)^2}{(a^2-b^2)^2}, \frac{(a+b)^2}{(a^2-b^2)^2}, \frac{a^2-b^2}{(a^2-b^2)^2}$.

7. $\frac{a}{5bc}, \frac{b}{10ac}, \frac{c}{3ab}$. *Ans.* $\frac{6a^2}{30abc}, \frac{3b^2}{30abc}, \frac{10c^2}{30abc}$.

8. $\frac{x}{x+y}, \frac{y}{x+y}, \frac{x}{x-y}, \frac{y}{x-y}$.

Ans. $\frac{x^2-xy}{x^2-y^2}, \frac{xy-y^2}{x^2-y^2}, \frac{x^2+xy}{x^2-y^2}, \frac{xy+y^2}{x^2-y^2}$.

9. $\frac{x}{x^2-y^2}, \frac{y}{x^3-y^3}$.

Ans. $\frac{x^3+x^2y+xy^2}{x^4+x^3y-xy^3-y^4}, \frac{xy+y^2}{x^4+x^3y-xy^3-y^4}$.

ADDITION OF FRACTIONS.

101. Definition.

Addition of fractions is the process of finding the sum of two or more fractions.

102. Rule.

1. *Reduce the fractions to equivalent fractions having a common denominator.*

2. *Add their numerators, and write their sum over the common denominator.*

103. Examples.

1. $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad}{bd}+\dfrac{bc}{bd}=\dfrac{ad+bc}{bd}.$

2. Add $\dfrac{a}{b}$, $\dfrac{b}{a}$, $\dfrac{b}{c}$ and $\dfrac{c}{b}$. *Ans.* $\dfrac{a^2c+b^2c+ab^2+ac^2}{abc}.$

3. Add $\dfrac{x}{x+5}$ and $\dfrac{x}{x-5}$. *Ans.* $\dfrac{2x^2}{x^2-25}.$

4. Add $\dfrac{x}{x-5}$ and $\dfrac{x}{x-7}$. *Ans.* $\dfrac{2x^2-12x}{x^2-12x+35}.$

5. Add $\dfrac{x-y}{x+y}$ and $\dfrac{x+y}{x-y}$. *Ans.* $\dfrac{2x^2+2y^2}{x^2-y^2}.$

6. Add $\dfrac{1}{1+x^2}$ and $\dfrac{1}{1-x^2}$. *Ans.* $\dfrac{2}{1-x^4}.$

7. Add $\dfrac{a}{a+b}$ and $\dfrac{b}{a-b}$. *Ans.* $\dfrac{a^2+b^2}{a^2-b^2}.$

8. Add $\dfrac{(a-b)^2}{(a+b)^2}$ and $\dfrac{(a+b)^2}{(a-b)^2}$.

Ans. $\dfrac{2a^4+12a^2b^2+2b^4}{a^4-2a^2b^2+b^4}.$

9. Add $\dfrac{a}{a-b}$, $\dfrac{b}{b-c}$ and $\dfrac{c}{c-d}$.

Ans. $\dfrac{3abc-2ac^2-2b^2c-2abd+acd+b^2d+bc^2}{abc-b^2c-ac^2+bc^2-abd+b^2d+acd-bcd}.$

10. Add $\dfrac{x}{x^2-7x+12}$, $\dfrac{x}{x^2-8x+15}$ and $\dfrac{x}{x^2-9x+20}$.

Ans. $\dfrac{3x}{x^2-8x+15}$.

SUBTRACTION OF FRACTIONS.

104. Definition.

Subtraction of fractions is the process of finding the difference of two fractions.

105. Rule.

1. *Reduce the fractions to equivalent fractions having a common denominator.*

2. *Subtract the numerator of the subtrahend from the numerator of the minuend, and write the difference over the common denominator.*

106. Examples.

1. $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{ad}{bd}-\dfrac{bc}{bd}=\dfrac{ad-bc}{bd}=\dfrac{a(d-c)-c(b-a)}{bd}$.

Scholium.—The last form is useful in solving numerical examples.

2. $\dfrac{21}{24}-\dfrac{33}{39}=\dfrac{21\times 6-33\times 3}{24\times 39}=\dfrac{27}{963}$.

3. From $\dfrac{65}{67}a$ take $\dfrac{73}{79}a$. *Ans.* $\dfrac{244}{5293}a$.

4. From $\dfrac{x}{x-7}$ take $\dfrac{x}{x-5}$. *Ans.* $\dfrac{2x}{x^2-12x+35}$.

5. From $\dfrac{x+y}{x-y}$ take $\dfrac{x-y}{x+y}$. *Ans.* $\dfrac{4xy}{x^2-y^2}$.

6. From $\dfrac{(a+b)^2}{(a-b)^2}$ take $\dfrac{(a-b)^2}{(a+b)^2}$.

Ans. $\dfrac{8ab\,(a^2+b^2)}{a^4-2a^2b^2+b^4}$.

7. From $\dfrac{a+b}{a-b}$ take $\dfrac{c-d}{c+d}$.

Ans. $\dfrac{2bc+2ad}{ac-bc+ad-bd}$.

8. From $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}$ take $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

Ans. $\dfrac{2\,(a-b)}{a+b}$.

9. From $\dfrac{x+5}{x^2+7x+12}$ take $\dfrac{x+3}{x^2+9x+20}$.

Ans. $\dfrac{4}{x+8x+15}$.

10. From $\left(\dfrac{a+b}{a-b}\right)^3$ take $\left(\dfrac{a-b}{a+b}\right)^3$.

Ans. $\dfrac{12a^5b+40a^3b^3+12ab^5}{a^6-3a^4b^2+3a^2b^4-b^6}$.

MULTIPLICATION OF FRACTIONS.

107. Definition.

Multiplication of fractions is the process of finding the product of two or more fractions.

108. Illustration.

$$\frac{5}{7}\times\frac{2}{3}=\frac{2}{3}\text{ of }\frac{5}{7};\quad\text{but }\frac{1}{3}\text{ of }\frac{5}{7}=\frac{5}{21};\ \therefore\ \frac{2}{3}\text{ of }\frac{5}{7}=\frac{10}{21}.$$

109. Rule.

Multiply the numerators together for the numerator, and the denominators for the denominator, cancelling common factors.

110. Examples.

1. $\frac{a^2-2ab+b^2}{a^2+2ab+b^2}\times\frac{a+b}{a-b}=\frac{(a-b)(a-b)(a+b)}{(a+b)(a+b)(a-b)}=\frac{a-b}{a+b}.$

2. Multiply $\frac{a}{a^2-b^2}$ by $\frac{a-b}{a+b}$. *Ans.* $\frac{a}{a^2+2ab+b^2}$.

3. Multiply $\frac{a}{b}+\frac{b}{a}$ by $\frac{a}{b}-\frac{b}{a}$. *Ans.* $\frac{a^4-b^4}{a^2b^2}$.

4. Multiply $\frac{x^3-y^3}{x^3+y^3}$ by $\frac{x+y}{x-y}$. *Ans.* $\frac{x^2+xy+y^2}{x^2-xy+y^2}$.

5. Multiply $\frac{x^4-y^4}{a^2+2ab+b^2}$ by $\frac{a+b}{x^3-x^2y+xy^2-y^3}$.

Ans. $\frac{x+y}{a+b}$.

6. Multiply $\frac{x^2-12x+35}{x^2-14x+48}$ by $\frac{x^2-15x+56}{x^2-11x+30}$.

Ans. $\frac{x^2-14x+49}{x^2-12x+36}$.

7. Multiply $1+\frac{a-b}{a+b}$ by $1-\frac{a-b}{a+b}$.

Ans. $\frac{4ab}{(a+b)^2}$.

8. Multiply $\frac{a+b}{a-b}+\frac{a-b}{a+b}$ by $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

Ans. $\frac{8ab(a^2+b^2)}{(a^2-b^2)^2}$.

9. Multiply $\dfrac{(a-b)^2}{(a+b)^2}$ by $\dfrac{a^3+b^3}{(a-b)^3}$.

Ans. $\dfrac{a^2-ab+b^2}{a^2-b^2}$.

10. Multiply $\dfrac{x^4-18x^2+80}{x^4+18x^2+80}$ by $\dfrac{x^4+2x^2-80}{x^4-2x^2-80}$.

Ans. $\dfrac{(x^2-8)^2}{(x^2+8)^2}$.

DIVISION OF FRACTIONS.

111. Definition.

Division of fractions is the process of finding the quotient of one fraction by another.

112. Illustration.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}, \text{ since } \frac{a}{b} \times \frac{d}{c} \times \frac{c}{d} = \frac{a}{b}.$$

113. Rule.

Multiply the dividend by the divisor inverted.

114. Examples.

1. $\dfrac{a^2-b^2}{a^2+2ab+b^2} \div \dfrac{(a+b)^2}{a^2-b^2}$

$$= \frac{(a+b)(a-b)}{(a+b)(a+b)} \times \frac{(a+b)(a-b)}{(a+b)(a+b)} = \frac{(a-b)^2}{(a+b)^2}.$$

2. Divide $\dfrac{a}{a^2-b^2}$ by $\dfrac{a-b}{a+b}$. *Ans.* $\dfrac{a}{(a-b)^2}$.

3. Divide $\dfrac{a^4-b^4}{a^2b^2}$ by $\dfrac{a}{b}-\dfrac{b}{a}$. *Ans.* $\dfrac{a^2+b^2}{ab}$.

4. Divide $\frac{x^3+y^3}{x^2-y^2}$ by $\frac{x+y}{x-y}$. *Ans.* $\frac{x^2-xy+y^2}{x+y}$.

5. Divide $\frac{x^2+7x+12}{x^2-11x+30}$ by $\frac{x^2-2x-24}{x^2+2x-35}$.

Ans. $\frac{x^2+10x+21}{x^2-12x+36}$.

6. Divide $\frac{a^4-b^4}{(a-b)^2}$ by $\frac{a^2+b^2}{a^2-b^2}$.

Ans. $a^2+2ab+b^2$.

7. Divide $1+\frac{a-b}{a+b}$ by $1-\frac{a-b}{a+b}$. *Ans.* $\frac{a}{b}$.

8. Divide $\frac{a+b}{a-b}-\frac{a-b}{a+b}$ by $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

Ans. $\frac{2ab}{a^2+b^2}$.

9. Divide $\frac{(x^2-8)^2}{(x^2+8)^2}$ by $\frac{x^4+2x^2-80}{x^4-2x^2-80}$.

Ans. $\frac{x^4-18x^2+80}{x^4+18x^2+80}$.

10. Divide $\frac{x^6-y^6}{(x-y)^2}$ by $\frac{x^3+y^3}{(x+y)(x-y)^3}$.

Ans. $x^5-x^3y^2-x^2y^3+y^5$.

115. Miscellaneous Examples.

1. $\frac{1}{a+\frac{1}{b}}=\frac{b}{ab+1}$. By multiplying both terms by b.

2. Reduce $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}$. *Ans.* $\frac{bc+1}{(ab+1)c+a}$.

3. $\dfrac{a+\dfrac{b}{cd}}{b+\dfrac{c}{bd}}=\dfrac{abcd+b^2}{b^2cd+c^2}$. By multiplying both terms by bcd, the l. c. m. of cd and bd.

4. Reduce $\dfrac{a-b+\dfrac{(a+b)^2}{a-b}}{a+b-\dfrac{(a-b)^2}{a+b}}$. *Ans.* $\dfrac{a^3+a^2b+ab^2+b^3}{2a^2b-2ab^2}$.

5. Reduce $\dfrac{x^4-y^4}{x^6-y^6}$. *Ans.* $\dfrac{x^2+y^2}{x^4+x^2y^2+y^4}$.

6. Reduce $\dfrac{x^{10}-y^{10}}{(x+y)(x^5-y^5)}$.

Ans. $x^4-x^3y+x^2y^2-xy^3+y^4$.

7. Reduce $x^2-xy+y^2+\dfrac{3x^2y+3xy^2}{x+y}$.

Ans. $(x+y)^2$.

8. Reduce $\dfrac{(a-b)^2}{(a+b)^2}\div\dfrac{a^2-b^2}{(a+b)^3}+\dfrac{a^3+a^2b-ab^2-b^3}{(a+b)^2}$.

Ans. $2(a-b)$.

9. From $\dfrac{x+5}{x+6}\times\dfrac{x-5}{x-6}$ take $\dfrac{x^2+25}{x^2+36}$.

Ans. $\dfrac{22x^2}{x^4-1296}$.

10. Reduce $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$. *Ans.* $\dfrac{adf+ae}{bdf+be+cf}$.

11. Reduce $\left(\dfrac{x}{x+y}+\dfrac{y}{x-y}\right)\div\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)$.

Ans. 1.

12. Add $\frac{ab}{(b-c)(c-a)}$, $\frac{ac}{(a-b)(b-c)}$ and $\frac{bc}{(a-b)(c-a)}$.

Ans. -1.

13. Multiply $\frac{(a+b)^3}{x^3-y^3}$ by $\frac{x^2+xy+y^2}{(a+b)^2}$ *Ans.* $\frac{a+b}{x-y}$.

14. Divide $\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}$ by $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

Ans. $\frac{ab}{a^2+b^2}$.

15. Reduce $\left(\frac{ax-a^2}{bx-b^2} \div \frac{x^2-a^2}{x^2-b^2}\right) \div \frac{bx+b^2}{ax+a^2}$.

Ans. $\frac{a^2}{b^2}$.

16. Divide $\frac{x^8-y^8}{x^6-y^6}$ by $\frac{x^4+y^4}{x^3+y^3}$.

Ans. $\frac{x^3+x^2y+xy^2+y^3}{x^2+xy+y^2}$.

17. Multiply $\frac{b^2x^4-b^6}{x^2-2bx+b^2}$ by $\frac{x^2-b^2}{bx^2+b^3}$.

Ans. $b(x+b)^2$.

18. Add $\frac{1}{x(x-y)(x-z)}$, $\frac{1}{y(y-x)(y-z)}$

and $\frac{1}{z(z-x)(z-y)}$. *Ans.* $\frac{1}{xyz}$.

116. Propositions.

1. *If the same quantity be added to both terms of a fraction, the resulting fraction will be equal to, greater, or less than the given fraction, according as the numerator is equal to, less, or greater than the denominator.*

Let $\frac{a}{b}$ be the given fraction, and $\frac{a+n}{b+n}$ the resulting fraction. Reducing to a common denominator, we have

$$\frac{a}{b}=\frac{ab+an}{b^2+bn} \text{ and } \frac{a+n}{b+n}=\frac{ab+bn}{b^2+bn}.$$

$$\text{If } a=b,\ \frac{ab+bn}{b^2+bn}=\frac{ab+an}{b^2+bn}, \text{ or } \frac{a+n}{b+n}=\frac{a}{b}.$$

$$\text{If } a<b,\ \frac{ab+bn}{b^2+bn}>\frac{ab+an}{b^2+bn}, \text{ or } \frac{a+n}{b+n}>\frac{a}{b}.$$

$$\text{If } a>b,\ \frac{ab+bn}{b^2+bn}<\frac{ab+an}{b^2+bn}, \text{ or } \frac{a+n}{b+n}<\frac{a}{b}.$$

2. *If the same quantity be subtracted from both terms of a fraction, the resulting fraction will be equal to, less, or greater than the given fraction, according as the numerator is equal to, less, or greater than the denominator.*

Let $\frac{a}{b}$ be the given fraction, and $\frac{a-n}{b-n}$ the resulting fraction. Reducing to a common denominator, we have

$$\frac{a}{b}=\frac{ab-an}{b^2-bn} \text{ and } \frac{a-n}{b-n}=\frac{ab-bn}{b^2-bn}.$$

$$\text{If } a=b,\ \frac{ab-bn}{b^2-bn}=\frac{ab-an}{b^2-bn}, \text{ or } \frac{a-n}{b-n}=\frac{a}{b}.$$

$$\text{If } a<b,\ \frac{ab-bn}{b^2-bn}<\frac{ab-an}{b^2-bn}, \text{ or } \frac{a-n}{b-n}<\frac{a}{b}.$$

$$\text{If } a>b,\ \frac{ab-bn}{b^2-bn}>\frac{ab-an}{b^2-bn}, \text{ or } \frac{a-n}{b-n}>\frac{a}{b}.$$

117. Symbols for Zero.

$$0, \frac{0}{a}, \frac{a}{\infty}, \frac{0}{\infty}.$$

1. The ordinary symbol for zero is 0.

2. Zero divided by a finite quantity is zero.

For, since $0 = a \times 0$, dividing these equals by a, we have

$$\frac{0}{a} = 0.$$

3. A finite quantity divided by infinity is zero.

The fraction $\frac{a}{b}$ becomes less as b becomes greater, and if b is greater than any assignable quantity, $\frac{a}{b}$ is less than any assignable quantity. Hence,

$$\frac{a}{\infty} = 0.$$

4. Zero divided by infinity is zero.

Since $\frac{a}{\infty} = 0$, for a still stronger reason,

$$\frac{0}{\infty} = 0.$$

118. Symbols for Infinity.

$$\infty, \frac{a}{0}, \frac{\infty}{b}, \frac{\infty}{0}.$$

1. The ordinary symbol for infinity is ∞.

2. A finite quantity divided by zero is infinity.

The fraction $\frac{a}{b}$ becomes greater as b becomes less, and if b is less than any assignable quantity, $\frac{a}{b}$ is greater than any assignable quantity. Hence,

$$\frac{a}{0} = \infty.$$

3. Infinity divided by a finite quantity is infinity.

The fraction $\frac{a}{b}$ becomes greater as a becomes greater, and if a is greater than any assignable quantity, $\frac{a}{b}$ is greater than any assignable quantity. Hence,

$$\frac{\infty}{b} = \infty.$$

4. Infinity divided by zero is infinity.

Since $\frac{\infty}{b} = \infty$, for a still stronger reason,

$$\frac{\infty}{0} = \infty.$$

119. Symbols for Indetermination.

$$\frac{0}{0}, \quad \infty \times 0, \quad \frac{\infty}{\infty}, \quad \infty - \infty.$$

1. Zero divided by zero is indeterminate.

Since $0 = a \times 0$, whatever be the value of a,

$$\frac{0}{0} = a, \text{ or any quantity whatever.}$$

2. Infinity multiplied by zero is indeterminate.

Since $\infty = \frac{a}{0}$, whatever be the value of a,

$$\infty \times 0 = a, \text{ or any quantity whatever.}$$

3. Infinity divided by infinity is indeterminate.

$$\frac{\infty}{\infty} = \infty \times \frac{1}{\infty} = \infty \times 0, \text{ which is indeterminate.}$$

4. $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$ becomes, if $a = 0$ and $b = 0$,

$$\infty - \infty = \frac{0}{0}, \text{ which is indeterminate.}$$

VANISHING FRACTIONS.

120. Definition.

A vanishing fraction is a fraction which, on a certain supposition, assumes the form of indetermination. Thus,

$$\frac{x^n - a^n}{x - a} \text{ assumes the form of } \frac{0}{0}, \text{ if } x = a.$$

A fraction which becomes $\frac{0}{0}$ for a particular supposition is not necessarily indeterminate; for the form of indetermination may appear in consequence of the existence of a common factor between the numerator and denominator, which factor becomes 0 for that supposition. Thus,

$$\frac{x^2 - a^2}{x - a} = \frac{0}{0}, \text{ if } x = a; \text{ but } \frac{x^2 - a^2}{x - a} = x + a = 2a, \text{ if } x = a.$$

To find the value of a vanishing fraction, we have the

121. Rule.

1. *Reduce the fraction to its lowest terms.*

2. *Make the supposition which would cause the original fraction to assume the form of indetermination, and the result will be the value of the fraction under that supposition.*

122. Examples.

1. Find the value of $\frac{x^3 - a^3}{x - a}$, when $x = a$.

Ans. $3a^2$.

2. Find the value of $\frac{x^4 - a^4}{x^2 - a^2}$, when $x = a$.

Ans. $2a^2$.

3. Find the value of $\frac{x^2 - a^2}{(x - a)^2}$, when $x = a$.

Ans. ∞.

4. Find the value of $\frac{a^3 - b^3}{a^2 - b^2}$, when $a = b$.

Ans. $\frac{3a}{2}$.

5. Find the value of $\frac{(x - a)^3}{x^3 - ax^2 - a^2x + a^3}$, when $x = a$.

Ans. 0.

6. Find the value of $\frac{x^n - a^n}{x - a}$, when $x = a$.

Ans. na^{n-1}.

7. Find the value of $\frac{x^2 - ax}{x^4 - 2ax^3 + 2a^3x - a^4}$, when $x = a$.

Ans. ∞.

8. Find the value of $\frac{a + x}{a - x} \times \frac{a^2 - x^2}{(a + x)^2}$, when $x = a$.

Ans. 1.

9. Find the value of $\frac{nx^{n+1} - (n + 1)x^n + 1}{1 - x^2}$, when $x = 1$.

Ans. 0.

10. Find the value of $\frac{x^{\frac{1}{2}} - n^{\frac{1}{2}} + (x - n)^{\frac{1}{2}}}{(x^2 - n^2)^{\frac{1}{2}}}$, when $x = n$.

Ans. $\frac{1}{\sqrt{2n}}$.

EQUATIONS OF THE FIRST DEGREE.

123. Definitions and Classification.

1. **An equation** is the expression of the equality of two quantities. Thus, $x = a$ is an equation.

2. **The members** of an equation are the two quantities connected by the sign of equality.

1st. *The first member* is the part on the left of the sign of equality.

2d. *The second member* is the part on the right of the sign of equality.

3. **An equation of the first degree** is an equation which involves only the first power of the unknown quantity and known quantities.

4. **A numerical equation** is an equation in which all of the known quantities are expressed by numbers.

5. **A literal equation** is an equation in which the known quantities are expressed, wholly or in part, by letters.

6. **An identical equation** is an equation in which the members are the same in form or in sense.

Thus, $x + a = x + a$ and $(x + a)^2 = x^2 + 2ax + a^2$ are identical.

7. Equations may involve one unknown quantity or more than one.

124. Formation of Equations.

1. Form an equation from $x + x$, if $x = 10$.

OPERATION.

$$\text{If } x = 10,\ x + x = 10 + 10,$$

$$\text{or } x + x = 20.$$

2. Form an equation from $x + x$, if $x = 3$.

3. Form an equation from $x + 2x$, if $x = 8$.

4. Form an equation from $x + 2x + 3x$, if $x = 5$.

5. Form an equation from $x + 3x + 5x$, if $x = 12$.

6. Form an equation from $\frac{x+2}{4} - \frac{x-1}{9}$, if $x = 10$.

7. Form an equation from $\frac{3x-5}{5} + \frac{2x+10}{4}$, if $x = 15$.

8. Form an equation from $\frac{x}{a} + \frac{x}{b}$, if $x = ab$.

9. Form an equation from $\frac{x}{a+b} + \frac{x}{a-b}$, if $x = a^2 - b^2$.

10. Form an equation from $\frac{x+ab}{a} - \frac{x-ab}{b}$, if $x = 2ab$.

TRANSFORMATION OF EQUATIONS.

125. Definition.

A transformation of an equation is any change in its form which does not affect the equality of its members.

126. First Transformation—Clearing of Fractions.

The object of this transformation is to make all the terms of the equation entire.

Clear the equation $\frac{3x}{4}+1=\frac{x}{2}+3$ of fractions.

OPERATION.

$$\frac{3x}{4}+1=\frac{x}{2}+3.$$
$$3x+4=2x+12.$$

Multiply both members by 4, the l. c. m. of the denominators, canceling the denominators. The members of the resulting equation will be equal. (Ax. 3.)

127. Rule.

Multiply both members of the equation by the l. c. m. of the denominators, reducing the fractions to entire quantities.

128. Examples.

Clear the following equations of fractions:

1. $\frac{3x+1}{3}+\frac{1}{6}=\frac{7x+3}{8}+\frac{1}{4}.$

Ans. $24x+8+4=21x+9+6.$

2. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=26.$ *Ans.* $6x+4x+3x=312.$

3. $\frac{5x-11}{4}-\frac{11x-1}{12}=\frac{x-1}{10}.$

Ans. $75x-165-55x+5=6x-6.$

4. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d.$ *Ans.* $bcx+acx+abx=abcd.$

5. $\frac{x+a}{a+b}=\frac{x-a}{a-b}.$

Ans. $ax+a^2-bx-ab=ax-a^2+bx-ab.$

6. $3x + 4x + 5x = \frac{1}{2}x - 4x - \frac{3x}{4} + \frac{a}{2}$.

Ans. $12x + 16x + 20x = 2x - 16x - 3x + 2a$.

7. $\frac{m}{x-2} + \frac{n}{x-3} = \frac{p}{x^2 - 5x + 6}$.

Ans. $mx - 3m + nx - 2n = p$.

8. $\frac{a}{x+a} + \frac{b}{x+b} = \frac{c}{x^2 + (a+b)x + ab}$.

Ans. $ax + ab + bx + ab = c$.

9. $\frac{x-a}{a+b} + \frac{x+b}{a-b} = \frac{p+q}{a^2 - b^2}$.

Ans. $ax - a^2 - bx + ab + ax + ab + bx + b^2 = p + q$.

10. $\frac{x+p}{p+q} - \frac{p-q}{x-p} = (p-q)(x+p)$.

Ans. $x^2 - p^2 - p^2 + q^2 = p^2x^2 - q^2x^2 - p^4 + p^2q^2$.

129. Second Transformation—Transposition.

The object of this transformation is to bring all of the unknown quantities into the first member of the equation, and all of the known quantities into the second.

Take the equation

$$(1) \qquad mx - q = p + nx.$$

If q be added to both members, and nx be subtracted from both members, the equality will not be destroyed. (Ax. 1, 2.)

Performing these operations, we have

$$(2) \qquad mx - nx = p + q.$$

Comparing (1) and (2), we find that $-q$ in the first member of (1) becomes $+q$ in the second member of (2), and that $+nx$ in the second member of (1) becomes $-nx$ in the first member of (2). Hence,

Any term may be transposed from one member of an equation to the other, if its sign be changed.

130. Rule.

Transpose all of the known quantities which are in the first member into the second, and all of the unknown quantities which are in the second member into the first, changing the signs of all the terms transposed.

131. Examples.

1. $ax - b = a + bx$. *Ans.* $ax - bx = a + b$.
2. $5x - 3 + x = 2x + 8$. *Ans.* $5x + x - 2x = 8 + 3$.
3. $7x + 4 - a = b - 3x$. *Ans.* $7x + 3x = a + b - 4$.
4. $(a + b)x - c = a + b - (a - b)x$.
   Ans. $(a + b)x + (a - b)x = a + b + c$.
5. $mx - nx - p = q + rx$.
   Ans. $mx - nx - rx = p + q$.

132. Third Transformation — Reduction.

The object of this transformation is to collect the terms as much as possible.

Thus, take the equation

$$5x + 3x = 30 - 6.$$

By reducing, we have

$$8x = 24.$$

Also, take the equation

$$mx - nx = p + q.$$

By factoring, we have

$$(m - n)\,x = p + q.$$

133. Rule.

Reduce both members as much as possible by adding and factoring.

134. Examples.

1. $9x - 5x = 9 + 7.$ *Ans.* $4x = 16.$

2. $8x + 7x = 25 + 5.$ *Ans.* $15x = 30.$

3. $ax + bx - a = b.$ *Ans.* $(a + b)\,x = a + b.$

4. $px - p = qx + 5p.$ *Ans.* $(p - q)\,x = 6p.$

5. $\left(\dfrac{a + b}{a - b}\right) x + x = \dfrac{a + b}{a - b} - 1.$ *Ans.* $2ax = 2b.$

135. Fourth Transformation — Dividing by the Co-efficient of x.

The object of this transformation is to find, from the reduced equation, the value of the unknown quantity.

Take the equation

$$8x = 24.$$

Dividing both members by 8, we have

$$x = 3. \quad [\text{Ax. } 4.]$$

Also, take the equation

$$(m-n)\,x = p + q.$$

Dividing both members by $m - n$, we have

$$x = \frac{p+q}{m-n}.$$

Take the equation

$$-3x = -12.$$

Dividing both members by -3, we have

$$x = 4.$$

Again, take the equation

$$-x = -a.$$

Dividing both members by -1, we have

$$x = a.$$

136. Rule.

Divide both members by the co-efficient of the unknown quantity.

137. Examples.

1. $(m-n)\,x = p - q$. *Ans.* $x = \dfrac{p-q}{m-n}$.
2. $6x - 5 = 3x + 10$. *Ans.* $x = 5$.
3. $\dfrac{4x}{9} - 8 = \dfrac{x}{3} - x + 2$. *Ans.* $x = 9$.
4. $\dfrac{ax}{b} - \dfrac{bx}{a} - c = d$. *Ans.* $x = \dfrac{ab\,(c+d)}{a^2 - b^2}$.
5. $\dfrac{x}{a} - (a-b)\,x - \dfrac{c}{d} = 1$. *Ans.* $x = \dfrac{a\,(c+d)}{d\,(1 - a^2 + ab)}$.

138. Verification.

The value of the unknown quantity is verified, if, when substituted for the unknown quantity, it satisfies the equation; that is, causes the two members to be equal.

Take the equation

$$(m - n)\,x = p + q.$$

$$\therefore \qquad x = \frac{p + q}{m - n}.$$

This value of x is thus verified:

$$(m - n)\,\frac{p + q}{m - n} = p + q.$$

$$\text{or,} \qquad p + q = p + q.$$

139. Rule.

Substitute the value of the unknown quantity for the unknown quantity in the given equation; and if it causes the two members to be equal, the value of the unknown quantity will be verified.

140. Examples.

1. $ax - bx = c - d.$ *Ans.* $x = \dfrac{c - d}{a - b}.$

2. $\dfrac{x}{4} - 5 = \dfrac{x}{6}.$ *Ans.* $x = 60.$

3. $\dfrac{x}{8} - \dfrac{x}{10} - 3 = 5.$ *Ans.* $x = 320.$

4. $\dfrac{x}{a} + \dfrac{x}{b} = c.$ *Ans.* $x = \dfrac{abc}{a + b}.$

5. $\dfrac{1}{3}x - \dfrac{1}{4}x - 7 = 8.$ *Ans.* $x = 180.$

SOLUTION OF EQUATIONS.

141. Definition.

The solution of an equation is the process of finding the value of the unknown quantity.

142. Rule.

Clear the equation of fractions, transpose, reduce, and divide by the co-efficient of the unknown quantity.

143. Examples.

1. $\frac{x}{2} - \frac{x}{4} - 2 = \frac{x}{5} - 1.$ *Ans.* $x = 20.$

2. $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + \frac{1}{2}.$ *Ans.* $x = \frac{6}{7}.$

3. $\frac{5x+3}{3} - \frac{3x-7}{2} = 5x - 10.$ *Ans.* $x = 3.$

4. $\frac{5x-1}{7} - \frac{9x-7}{5} = \frac{5-9x}{11}.$ *Ans.* $x = 3.$

5. $\frac{2x-6}{5} - \frac{3x}{13} - \frac{x-4}{9} = 0.$ *Ans.* $x = 13.$

6. $\frac{x}{4} - \frac{2x-9}{3} = \frac{5x+8}{6}.$ *Ans.* $x = 1\frac{1}{3}.$

7. $\frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14.$ *Ans.* $x = 7.$

8. $x - \frac{26-x}{2} = \frac{x-11}{3}.$ *Ans.* $x = 8.$

9. $\frac{x+1}{2} - \frac{5-x}{4} = 14 - \frac{x+2}{3}.$ *Ans.* $x = 13.$

10. $\frac{3x-11}{4} - \frac{28-9x}{8} = 4x - \frac{59}{4}.$ *Ans.* $x = 4.$

11. $\dfrac{2x-1}{3}-\dfrac{3x-2}{4}=\dfrac{5x-4}{6}-\dfrac{7x+6}{12}.$

Ans. $x=4.$

12. $(x+a)(x+b)=(x+c)(x+d).$

Ans. $x=\dfrac{cd-ab}{a+b-c-d}.$

13. $\dfrac{m}{x-a}=\dfrac{n}{x-b}.$ *Ans.* $x=\dfrac{bm-an}{m-n}.$

14. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}.$ *Ans.* $x=4\frac{1}{2}.$

15. $\dfrac{a}{x-2}-\dfrac{a}{x-4}=\dfrac{a}{x-6}-\dfrac{a}{x-8}.$ *Ans.* $x=5.$

16. $\dfrac{2x-9}{27}-\dfrac{x-3}{4}+\dfrac{x}{18}=\dfrac{25-3x}{3}.$ *Ans.* $x=9.$

17. $\dfrac{x}{m}-\dfrac{x}{n}+\dfrac{x}{p}-\dfrac{x}{q}=r.$

Ans. $x=\dfrac{mnpqr}{npq-mpq+mnq-mnp}.$

18. $(x+a)(x-a)=(x-a)(x-b)+ab+b^2.$

Ans. $x=a+b.$

19. $\dfrac{p+q}{x-r}=\dfrac{p}{x-p}+\dfrac{q}{x-q}.$ *Ans.* $x=\dfrac{p^2q-2pqr+pq^2}{p^2-pr-qr+q^2}.$

20. $\dfrac{x^3+b^3}{x+b}=x^2-ax+a^2.$ *Ans.* $x=a+b.$

21. $\dfrac{5x-1}{11}-\dfrac{11x-3}{12}-\dfrac{13x-15}{3}=\dfrac{24-61x}{35}-23.$

Ans. $x=9.$

22. $\dfrac{qx}{p}-\dfrac{s}{r}=\dfrac{p}{q}-\dfrac{rx}{s}.$ *Ans.* $x=\dfrac{ps}{qr}.$

23. $\dfrac{px}{q} - r = \dfrac{mx}{n} - s.$ *Ans.* $x = \dfrac{qnr - qns}{pn - qm}.$

24. $\dfrac{ax^m}{bx + c} = \dfrac{px^m}{qx + r}.$ *Ans.* $x = \dfrac{cp - ar}{aq - bp}.$

25. $\dfrac{(a+b)(x-b)}{a-b} - \dfrac{a^2 - bx}{b} + 2x = \dfrac{4ab - b^2}{a+b} + 3a.$

Ans. $x = \dfrac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{4a^2b + 2ab^2 - 2b^3}.$

THE ROOTS OF AN EQUATION.

144. Definition.

A root of an equation is a value which substituted for the unknown quantity will verify the equation.

145. Propositions.

1. *Every equation of the first degree, not identical, involving but one unknown quantity, has one root, and but one.*

By clearing of fractions, transposing and reducing, if necessary, every equation of the first degree will take the form

$$(1) \qquad ax = b.$$

$$\therefore \quad (2) \qquad x = \frac{b}{a}.$$

This value of x is thus verified:

$$a \times \frac{b}{a} = b.$$

$$\text{or,} \quad b = b.$$

Now, it is evident that any value for x greater than $\frac{b}{a}$ would cause the first member of equation (1) to be greater than the second, and that every value for x less than $\frac{b}{a}$ would cause the first member of (1) to be less than the second. Hence, there can be no root either greater or less than $\frac{b}{a}$; that is, the equation has but one root, and that root is $\frac{b}{a}$.

The above proposition can also be proved thus:

Take the equation

$$(1) \qquad ax = b.$$

Suppose the equation has two different roots, p and q. Then these roots will satisfy equation (1), and give

$$(2) \qquad ap = b.$$

$$(3) \qquad aq = b.$$

$$(2) - (3) = (4) \qquad a(p - q) = 0.$$

But this is impossible; for $p - q$ is not 0, since p and q are unequal by hypothesis, and a is not 0; hence, $a(p - q)$ can not be equal to 0, and the supposition that the equation has two roots, which led to this false conclusion, is false.

2. *If the second member of an equation be transposed into the first, the result is divisible by the unknown quantity, minus the root of the equation.*

Let $\qquad (1) \qquad ax = b.$

$$\therefore \quad (2) \qquad x = \frac{b}{a}.$$

Then, $\qquad (ax - b) \div \left(x - \frac{b}{a}\right) = a.$

146. Problems.

The solution of a problem consists of two parts:

1st. The statement, or the formation of the equation which shall express the relation between the known and the unknown quantities.

2d. The solution of the equation.

1. Find a number which is equal to the sum of its third, its fourth, and 10.

Let $x =$ the number. Then, by the conditions of the problem, we have

$$x = \frac{x}{3} + \frac{x}{4} + 10.$$

Clearing of fractions, we have

$$12x = 4x + 3x + 120.$$

Transposing, we have

$$12x - 4x - 3x = 120.$$

Reducing, we have

$$5x = 120.$$

Dividing by 5, we have

$$x = 24.$$

2. Find a number whose third part exceeds its fourth by 15. *Ans.* 180.

3. What number whose half, third, and fourth parts, together with 11, are equal to 50? *Ans.* 36.

4. Divide $16000 between A and B, so that A's share shall be to B's as 3 to 5.

Let $3x =$ A's share.

Then $5x =$ B's share.

Then $$3x + 5x = \$16000. \qquad [\text{Ax. } 8.]$$

Or $$8x = \$16000.$$

$$\therefore \quad x = \$2000.$$

$$3x = \$6000 \;= \text{A's share.}$$

$$5x = \$10000 = \text{B's share.}$$

5. Divide $18000 between A and B, so that A's share shall be to B's as 4 is to 5.

Ans. A's = $8000; B's = $10000.

6. Divide n into two parts, so that the first shall be to the second as p to q. *Ans.* $\frac{np}{p+q}$, $\frac{nq}{p+q}$.

Scholium.—This is a general problem of which the 4th and 5th are particular cases. The answers of the particular problems may be deduced from those of the general problem.

Thus, in the 4th problem, $n = \$16000$, $p = 3$, and $q = 5$. Then,

$$\text{A's share} = \frac{\$16000 \times 3}{3 + 5} = \$6000.$$

$$\text{B's share} = \frac{\$16000 \times 5}{3 + 5} = \$10000.$$

In like manner, let the answers of the 5th be deduced; and in all the following problems, let the answers of the particular problems be deduced from those of the general problem.

7. A left a certain town at the rate of 4 miles an hour, and in 12 hours was followed by B at the rate of 10 miles an hour. In how many hours did B overtake A? *Ans.* 8.

8. A left a certain town at the rate of a miles per hour, and in n hours was followed by B at the rate of b miles an hour. In how many hours did B overtake A?

Ans. $\frac{an}{b - a}$.

9. A man was hired for 50 days, on these conditions: for every day he worked he was to receive 75 cents, and for every day he was idle he was to pay 25 cents for his board. At the expiration of the time, he received \$27.50. How many days did he work, and how many days was he idle? *Ans.* Worked, 40; idle, 10.

10. A man was hired for n days, on these conditions: for every day he worked, he was to receive p cents, and for every day he was idle he was to pay q cents for his board. At the expiration of the time, he received a cents. How many days did he work, and how many days was he idle? *Ans.* Worked, $\frac{a+nq}{p+q}$; idle, $\frac{np-a}{p+q}$.

11. A can do a certain work in 10 days, B in 12 days, and C in 15 days. In what time can they together do the work? *Ans.* 4 days.

12. A can do a certain work in a days, B in b days, and C in c days. In what time can they together do the work? *Ans.* $\frac{abc}{bc+ac+ab}$ days.

13. A, who has 3 hours to spare, wishes to enjoy as long as possible the company of B, who leaves immediately in a coach which travels 8 miles per hour. How far can A ride so as to walk back in time, at the rate of 4 miles per hour? *Ans.* 8 miles.

14. A rode a certain distance at the rate of m miles per hour, and walked back at the rate of n miles per hour, and performed the whole journey in t hours. How far did he ride? *Ans.* $\frac{mnt}{m+n}$ miles.

15. The interest on $\frac{2}{5}$ of a certain capital, at 5 % + the interest on $\frac{3}{5}$ of it at 6 % is equal to \$840. Required the capital. *Ans.* \$15000.

16. The interest on $\frac{n}{d}$ of a certain capital, at r % + the interest on the remaining part, at r' % $= i$. Required the capital.

Ans. $\dfrac{100di}{nr + (d - n)\,r'}$.

17. A man rows a boat with the tide 7 miles in 40 minutes, and returns against a tide $\frac{1}{3}$ as strong in 1 hour. What is the rate of the strongest tide? *Ans.* $2\frac{5}{8}$ miles.

18. A man rows a boat with the tide m miles in t hours, and returns against a tide $\frac{1}{n}$ as strong in t' hours. Required the rate of the strongest tide.

Ans. $\dfrac{mn\,(t' - t)}{tt'\,(n + 1)}$ miles.

19. The difference of the squares of two consecutive numbers is 21. What are those numbers?

Ans. 10 and 11.

20. The difference of the squares of two consecutive numbers is d. What are those numbers?

Ans. $\dfrac{d - 1}{2}$ and $\dfrac{d + 1}{2}$.

21. The difference of two numbers is 5, and the difference of their squares is 45. What are those numbers?

Ans. 2 and 7.

22. The difference of two numbers is d, and the difference of their squares is d'. What are those numbers?

Ans. $\dfrac{d' - d^2}{2d}$ and $\dfrac{d' + d^2}{2d}$.

23. Divide 45 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to each other. *Ans.* 10, 15, 20.

24. Divide a into three such parts that $\frac{1}{m}$ of the first, $\frac{1}{n}$ of the second, and $\frac{1}{p}$ of the third shall be equal to each other. *Ans.* $\frac{am}{m+n+p}$, $\frac{an}{m+n+p}$, $\frac{ap}{m+n+p}$.

25. Divide 72 into four such parts that if the first be increased by 5, the second diminished by 5, the third multiplied by 5, and the fourth divided by 5, the results shall be equal. *Ans.* 5, 15, 2, 50.

26. Divide a into four such parts that if the first be increased by n, the second diminished by n, the third multiplied by n. and the fourth divided by n, the results shall be equal

Ans. $\frac{an}{(n+1)^2}-n$, $\frac{an}{(n+1)^2}+n$, $\frac{a}{(n+1)^2}$, $\frac{an^2}{(n+1)^2}$.

27. The greater of two numbers is three times the less; but if each be increased by 15, the greater will be twice the less. Required the numbers. *Ans.* 15, 45.

28. The greater of two numbers is m times the less; but if each be increased by p, the greater will be n times the less. Required the numbers.

Ans. $\frac{p\,(n-1)}{m-n}$, $\frac{mp\,(n-1)}{m-n}$.

29. A's money + 2 times the sum of B's and C's = \$8000; B's money + 3 times the sum of A's and C's = \$10500; C's money + 4 times the sum of A's and B's = \$12000. They together have \$4500. What has each?

Ans. A has \$1000; B, \$1500; C, \$2000.

30. A's money + l times the sum of B's and C's = p; B's money + m times the sum of A's and C's = q;

C's money + n times the sum of A's and B's $= r$. They together have s. What has each?

Ans. A has $\frac{p-ls}{1-l}$; B, $\frac{q-ms}{1-m}$; C, $\frac{r-ns}{1-n}$.

31. If 12 oxen eat the grass of $3\frac{1}{3}$ acres in 4 weeks, and 21 oxen eat the grass of 10 acres in 9 weeks, how many oxen will eat the grass of 24 acres in 18 weeks, the grass being at first equal on every acre, and growing uniformly?

Let $x =$ the growth on 1 *A.* in 1 *wk.*, the grass at first on 1 *A.* being regarded as the unit.

Then $\frac{40}{3}x =$ the growth on $3\frac{1}{3}$ *A.* in 4 *wk.*

$\frac{10+40x}{3} =$ the grass on $3\frac{1}{3}$ *A.* + the growth in 4 *wk.*

$\therefore\ \frac{10+40x}{3} =$ what 12 oxen eat in 4 *wk.*

$\therefore\ \frac{10+40x}{144} =$ what 1 ox eats in 1 *wk.*

$90x =$ the growth on 10 *A.* in 9 *wk.*

$10+90x =$ the grass on 10 *A.* + the growth in 9 *wk.*

$\therefore\ 10+90x =$ what 21 oxen eat in 9 *wk.*

$\therefore\ \frac{10+90x}{189} =$ what 1 ox eats in 1 *wk.*

$$\therefore\ \frac{10+90x}{189} = \frac{10+40x}{144}$$

$$\therefore\ x = \frac{1}{12}.$$

$\therefore\ \frac{10+40x}{144} = \frac{5}{54} =$ what 1 ox eats in 1 *wk.*

$\therefore\ \frac{5}{54} \times 18 = \frac{5}{3} =$ what 1 ox eats in 18 *wk.*

$\frac{1}{12} \times \frac{24}{1} \times \frac{18}{1} = 36 =$ the growth on 24 *A.* in 18 *wk.*

$24 + 36 = 60 =$ the grass on 24 *A.* + the growth in 18 *wk.*

$60 \div \frac{5}{3} = 36 =$ the number of oxen that eat the grass on 24 *A.* + the growth in 18 *wk.*

32. If a oxen eat m acres of grass in q weeks, and b oxen eat n acres in r weeks, how many oxen will eat p acres in s weeks, the grass being at first equal on every acre, and growing uniformly? *Ans.* $\frac{anpqr - bmpqr + bmprs - anpqs}{mnrs - mnqs}$.

ELIMINATION.

147. Definition.

Elimination is the process of combining equations involving two or more unknown quantities so as to cause the disappearance of one or more of the unknown quantities.

ELIMINATION BY ADDITION OR SUBTRACTION.

148. Illustrations.

1. Given $\left\{\begin{array}{ll}(1) & x + y = s. \\ (2) & x - y = d.\end{array}\right\}$ Required x and y.

SOLUTION.

$(1) + (2) = (3) \quad 2x = s + d. \quad \therefore \quad x = \frac{1}{2}s + \frac{1}{2}d.$

$(1) - (2) = (4) \quad 2y = s - d. \quad \therefore \quad y = \frac{1}{2}s - \frac{1}{2}d.$

2. Given $\left\{\begin{array}{ll}(1) & 2x + 7y = 29. \\ (2) & 3x + 5y = 27.\end{array}\right\}$ Required x and y.

SOLUTION.

$$\begin{array}{ll}(1) \times 3 = (3) & 6x + 21y = 87. \\ (2) \times 2 = (4) & 6x + 10y = 54. \\ (3) - (4) = (5) & 11y = 33. \quad [\text{Ax. } 2.]\end{array}$$

$$\therefore \quad y = 3.$$

Substituting this value of y for y in (1), we have

$$2x + 21 = 29.$$

Transposing and reducing, we have

$$2x = 8.$$

$$\therefore \quad x = 4.$$

3. Given $\left\{\begin{array}{ll}(1) & 5x + 6y = 49. \\ (2) & 7x - 4y = 19.\end{array}\right\}$ Required x and y.

SOLUTION.

$$\begin{array}{ll}(1) \times 2 = (3) & 10x + 12y = 98. \\ (2) \times 3 = (4) & 21x - 12y = 57. \\ (3) + (4) = (5) & 31x = 155.\end{array}$$

$$\therefore \quad x = 5.$$

This value of x substituted in (1) or (2) will give

$$y = 4.$$

4. Given $\left\{\begin{array}{ll}(1) & 6x + 5y = 61. \\ (2) & 3x + 4y = 38.\end{array}\right\}$ Required x and y.

SOLUTION.

$$\begin{array}{rl}(2) \times 2 = (3) & 6x + 8y = 76. \\ (1) & 6x + 5y = 61. \\ (3) - (1) = (4) & 3y = 15. \\ \therefore & y = 5.\end{array}$$

Substituting in (1) or (2), we find

$$x = 6.$$

5. Given $\left\{\begin{array}{ll}(1) & 8x + 6y = 64. \\ (2) & 5x - 3y = 13.\end{array}\right\}$ Required x and y.

SOLUTION.

$$\begin{array}{rl}(1) \div 2 = (3) & 4x + 3y = 32. \\ (2) & 5x - 3y = 13. \\ (3) + (2) = (4) & 9x = 45. \\ \therefore & x = 5.\end{array}$$

Substituting in (1), (2), or (3), we find

$$y = 4.$$

149. Rule.

1. *Prepare the equations, if necessary, either by multiplication or division, so that the co-efficients of the quantity to be eliminated shall be numerically equal.*

2. *Then, if these co-efficients have unlike signs, add the equations; but if they have like signs, subtract one equation from the other.*

3. *Find the value of the unknown quantity in the resulting equation.*

4. *Substitute the value of the unknown quantity found in one of the equations involving two unknown quantities, and find the value of the other unknown quantity.*

150. Examples.

1. Given $\left\{\begin{array}{l} 3x + 5y = 19. \\ 4x - 2y = 8. \end{array}\right\}$ Required x and y.

Ans. $x = 3,\ y = 2.$

2. Given $\left\{\begin{array}{l} 5x + 11y = 69. \\ 7x - 5y = 15. \end{array}\right\}$ Required x and y.

Ans. $x = 5,\ y = 4.$

3. Given $\left\{\begin{array}{l} 9x + 6y = 39. \\ 5x + 3y = 21. \end{array}\right\}$ Required x and y.

Ans. $x = 3,\ y = 2.$

4. Given $\left\{\begin{array}{l} 4x + 6y = 58. \\ 7x - 3y = 34. \end{array}\right\}$ Required x and y.

Ans. $x = 7,\ y = 5.$

5. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 6. \\ \frac{1}{4}x + \frac{1}{2}y = 5. \end{array}\right\}$ Required x and y.

Ans. $x = 8,\ y = 6.$

6. Given $\left\{\begin{array}{l} \frac{1}{3}x - \frac{1}{5}y = 4. \\ \frac{1}{7}x + \frac{1}{3}y = 8. \end{array}\right\}$ Required x and y.

Ans. $x = 21,\ y = 15.$

7. Given $\begin{cases} ax + by = m. \\ cx + dy = n. \end{cases}$ Required x and y.

$$Ans.\ x = \frac{dm - bn}{ad - bc},\ y = \frac{cm - an}{bc - ad}.$$

8. Given $\begin{cases} mx - ny = r. \\ px - qy = s. \end{cases}$ Required x and y.

$$Ans.\ x = \frac{qr - ns}{mq - np},\ y = \frac{pr - ms}{mq - np}.$$

9. Given $\begin{cases} x + y = s. \\ \dfrac{x}{a} + \dfrac{y}{b} = q. \end{cases}$ Required x and y.

$$Ans.\ x = \frac{a(s - bq)}{a - b},\ y = \frac{b(aq - s)}{a - b}.$$

10. Given $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a. \\ \dfrac{1}{x} - \dfrac{1}{y} = b. \end{cases}$ Required x and y.

$$Ans.\ x = \frac{2}{a + b},\ y = \frac{2}{a - b}.$$

ELIMINATION BY SUBSTITUTION.

151. Illustration.

Given $\begin{cases} (1) \quad 2x + 3y = 19. \\ (2) \quad 3x + 2y = 21. \end{cases}$ Required x and y.

SOLUTION.

By transposing $2x$ in (1), we have

$$3y = 19 - 2x.$$

$$\therefore \quad (3) \quad y = \frac{19 - 2x}{3}.$$

Substituting this value of y in (2), we have

$$3x + 2\left(\frac{19 - 2x}{3}\right) = 21.$$

Clearing of fractions, and developing,

$$9x + 38 - 4x = 63.$$

Transposing and reducing,

$$5x = 25.$$

$$\therefore \quad x = 5.$$

Substituting in (3), we have

$$y = \frac{19 - 10}{3} = 3.$$

152. Rule.

1. *Find from one equation the value of one unknown quantity in terms of the known and other unknown quantities in that equation.*

2. *Substitute this value in the other equation for the unknown quantity which it represents.*

3. *Find the value of the unknown quantity in the resulting equation.*

4. *Substitute the value of the unknown quantity thus found in the expression for the other unknown quantity, and reduce.*

153. Examples.

1. Given $\left\{\begin{array}{l} x + 2y = 13. \\ 2x + y = 14. \end{array}\right\}$ Required x and y.

   Ans. $x = 5$, $y = 4$.

2. Given $\left\{\begin{array}{l} 4x + 3y = 20. \\ 2x + 6y = 25. \end{array}\right\}$ Required x and y.

   Ans. $x = 2\frac{1}{2}$, $y = 3\frac{1}{3}$.

3. Given $\left\{\begin{array}{l} 7x + 9y = 130. \\ 9x + 7y = 126. \end{array}\right\}$ Required x and y.

   Ans. $x = 7$, $y = 9$.

4. Given $\left\{\begin{array}{l} 11x - 13y = 0. \\ 13x - 11y = 48. \end{array}\right\}$ Required x and y.

   Ans. $x = 13$, $y = 11$.

5. Given $\left\{\begin{array}{l} x + y = s. \\ x - y = d. \end{array}\right\}$ Required x and y.

   Ans. $x = \frac{1}{2}s + \frac{1}{2}d$, $y = \frac{1}{2}s - \frac{1}{2}d$.

6. Given $\left\{\begin{array}{l} mx + ny = a. \\ px + qy = b. \end{array}\right\}$ Required x and y.

   Ans. $x = \dfrac{aq - bn}{mq - np}$, $y = \dfrac{bm - ap}{mq - np}$.

7. Given $\left\{\begin{array}{l} ax + by = c. \\ mx = ny. \end{array}\right\}$ Required x and y.

   Ans. $x = \dfrac{cn}{an + bm}$, $y = \dfrac{cm}{an + bm}$.

8. Given $\left\{ \begin{array}{l} \dfrac{x}{m} + \dfrac{y}{n} = a. \\ \\ \dfrac{x}{p} + \dfrac{y}{q} = b. \end{array} \right\}$ Required x and y.

Ans. $x = \dfrac{amnp - bmpq}{np - mq}$, $y = \dfrac{bnpq - amnq}{np - mq}$.

9. Given $\left\{ \begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = c. \\ \\ \dfrac{x}{m} = \dfrac{y}{n}. \end{array} \right\}$ Required x and y.

Ans. $x = \dfrac{abcm}{an + bm}$, $y = \dfrac{abcn}{an + bm}$.

10. Given $\left\{ \begin{array}{l} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \\ \dfrac{c}{x} + \dfrac{d}{y} = n. \end{array} \right\}$ Required x and y.

Ans. $x = \dfrac{ad - bc}{dm - bn}$, $y = \dfrac{ad - bc}{an - cm}$.

ELIMINATION BY COMPARISON.

154. Illustration.

Given $\left\{ \begin{array}{ll} (1) & 4x + 3y = 27. \\ (2) & 3x + 5y = 34. \end{array} \right\}$ Required x and y.

SOLUTION.

From (1) we find (3) $y = \dfrac{27 - 4x}{3}$.

From (2) we find (4) $y = \dfrac{34 - 3x}{5}$.

$$\therefore \text{ by Ax. 9, } \quad \frac{34 - 3x}{5} = \frac{27 - 4x}{3}.$$

Clearing of fractions, we have

$$102 - 9x = 135 - 20x.$$

Transposing and reducing, we have

$$11x = 33.$$

$$\therefore \quad x = 3.$$

Substituting in (3) or (4), we find

$$y = 5.$$

155. Rule.

1. *Find from each equation the value of the same unknown quantity, in terms of the known and other unknown quantities in that equation.*

2. *Write these values equal to each other.*

3. *Find the value of the unknown quantity in the resulting equation.*

4. *Substitute the value of the unknown quantity thus found in either expression for the other unknown quantity, and reduce.*

156. Examples.

1. Given $\left\{\begin{array}{l} 7x + 5y = 74. \\ 5x + 7y = 70. \end{array}\right\}$ Required x and y.

Ans. $x = 7, \; y = 5.$

2. Given $\left\{\begin{array}{l} 13x + 17y = 304. \\ 17x + 19y = 362. \end{array}\right\}$ Required x and y.

Ans. $x = 9$, $y = 11$.

3. Given $\left\{\begin{array}{l} 9x - 11y = 0. \\ 12x - 10y = 42. \end{array}\right\}$ Required x and y.

Ans. $x = 11$, $y = 9$.

4. Given $\left\{\begin{array}{l} 100x - 50y = 500. \\ 150x - 100y = 500. \end{array}\right\}$ Required x and y.

Ans. $x = 10$, $y = 10$.

5. Given $\left\{\begin{array}{l} \dfrac{x}{7} + 7y = 99. \\ \dfrac{y}{7} + 7x = 51. \end{array}\right\}$ Required x and y.

Ans. $x = 7$, $y = 14$.

6. Given $\left\{\begin{array}{l} \dfrac{3}{4}x + \dfrac{2}{3}y = 15. \\ \dfrac{5}{6}x + \dfrac{7}{9}y = 17. \end{array}\right\}$ Required x and y.

Ans. $x = 12$, $y = 9$.

7. Given $\left\{\begin{array}{l} mx + by = p. \\ nx + cy = q. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{cp - bq}{cm - bn}$, $y = \dfrac{mq - np}{cm - bn}$.

8. Given $\left\{\begin{array}{l} mx = ny. \\ y = ax + b. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{bn}{m - an}$, $y = \dfrac{bm}{m - an}$.

9. Given $\left\{\begin{array}{l} ax + by = p. \\ ax + dy = q. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{bq - pd}{a(b - d)}, \quad y = \dfrac{p - q}{b - d}.$

10. Given $\left\{\begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = a. \\ \\ \dfrac{n}{x} + \dfrac{m}{y} = b. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{m^2 - n^2}{am - bn}, \quad y = \dfrac{n^2 - m^2}{an - bm}$

ELIMINATION BY INDETERMINATE MULTIPLIERS.

157. Illustration.

Given $\left\{\begin{array}{ll} (1) & 5x + 3y = 34. \\ (2) & 3x + 5y = 30. \end{array}\right\}$ Required x and y.

SOLUTION.

$$(1) \times m = (3) \quad 5mx + 3my = 34m.$$

$$(2) \quad 3x + 5y = 30.$$

$$(3) - (2) = (4) \quad (5m - 3)\,x + (3m - 5)\,y = 34m - 30.$$

1st. Assume

$$3m - 5 = 0.$$

$$\therefore \quad m = \frac{5}{3},$$

and (4) becomes

$$\frac{16}{3}\,x = \frac{80}{3}.$$

$$\therefore \quad x = 5.$$

2d. Assume

$$5m - 3 = 0.$$

$$\therefore \quad m = \frac{3}{5},$$

and (4) becomes

$$-\frac{16}{5}y = -\frac{48}{5}.$$

$$\therefore \quad y = 3.$$

158. Rule.

1. *Multiply either equation by* m, *and from the resulting equation subtract the other equation.*

2. *Assume the co-efficient of one unknown quantity in the resulting equation equal to* 0, *from which deduce the value of* m *which substitute in the preceding equation, and reduce.*

159. Examples.

1. Given $\left\{\begin{array}{r} 4x + 5y = 41. \\ 5x + 4y = 40. \end{array}\right\}$ Required x and y.

Ans. $x = 4$, $y = 5$.

2. Given $\left\{\begin{array}{r} 5x + 7y = 50. \\ 6x + 5y = 43. \end{array}\right\}$ Required x and y.

Ans. $x = 3$, $y = 5$.

3. Given $\left\{\begin{array}{r} 8x + 9y = 43. \\ 11x + 13y = 61. \end{array}\right\}$ Required x and y.

Ans. $x = 2$, $y = 3$.

4. Given $\left\{\begin{array}{l} 12x + 7y = 88. \\ 7x - 5y = 15. \end{array}\right\}$ Required x and y.

Ans. $x = 5,\ y = 4.$

5. Given $\left\{\begin{array}{l} ax + by = p. \\ cx + dy = q. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{dp - bq}{ad - bc},\ \ y = \dfrac{aq - cp}{ad - bc}.$

ELIMINATION BY THE GREATEST COMMON DIVISOR.

160. Illustration.

Given $\left\{\begin{array}{ll} (1) & 4x + 5y - 41 = 0. \\ (2) & 5x + 4y - 40 = 0. \end{array}\right\}$ Required x and y.

If the value of y were found and substituted in these two equations, we should have two equations involving x and giving the same value for x. Since the first members of these equations are divisible by x minus the value of x (Art. 145, Prop. 2), they must have a common divisor involving x. Let us now proceed as in finding the g. c. d., and put the final remainder involving y equal to 0, which must be the case since the equations have a common divisor.

Multiplying (1) by 5, and dividing the result by (2), and writing the remainder equal to 0, we have

$$\begin{array}{r|l} 20x + 25y - 205 & \underline{5x + 4y - 40} \\ \underline{20x + 16y - 160} & 4 \\ 9y - 45 = 0 & \end{array}$$

$$\therefore\quad y = 5.$$

161. Rule.

Transpose the second members, proceed with the resulting first members as in finding the greatest common divisor, and place the remainder involving one unknown quantity equal to 0, which will give the value of that unknown quantity.

162. Examples.

1. Given $\left\{\begin{array}{l} 3x + 4y = 39. \\ 5x + 6y = 61. \end{array}\right\}$ Required x and y.

Ans. $x = 5, \ y = 6.$

2. Given $\left\{\begin{array}{l} 3x + 7y = 42. \\ 7x + 3y = 58. \end{array}\right\}$ Required x and y.

Ans. $x = 7, \ y = 3$

3. Given $\left\{\begin{array}{l} 5x + 7y = 184. \\ 11x + 9y = 296. \end{array}\right\}$ Required x and y.

Ans. $x = 13, \ y = 17.$

4. Given $\left\{\begin{array}{l} 3x - 2y = 0. \\ 5x - 4y = -20. \end{array}\right\}$ Required x and y.

Ans. $x = 20, \ y = 30.$

5. Given $\left\{\begin{array}{l} mx + ny = a. \\ px + qy = b. \end{array}\right\}$ Required x and y.

Ans. $x = \dfrac{aq - bn}{mq - np}, \quad y = \dfrac{bm - ap}{mq - np}.$

EQUATIONS INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

163. Illustration.

$$\text{Given}\left\{\begin{array}{lr} (1) & 3x + 2y + 5z = 32. \\ (2) & 4x - 3y + 6z = 23. \\ (3) & 6x + 5y - 8z = -5. \end{array}\right\} \text{Required } x,\ y, \text{ and } z.$$

SOLUTION.

$$\begin{array}{lr} (1) \times 4 = (4) & 12x + 8y + 20z = 128. \\ (2) \times 3 = (5) & 12x - 9y + 18z = 69. \\ (4) - (5) = (6) & 17y + 2z = 59. \\ (1) \times 2 = (7) & 6x + 4y + 10z = 64. \\ (3) & 6x + 5y - 8z = -5. \\ (7) - (3) = (8) & -y + 18z = 69. \\ (6) \times 9 = (9) & 153y + 18z = 531. \\ (9) - (8) = (10) & 154y = 462. \end{array}$$

$$\therefore\ y = 3.$$

Substituting the value of y in (8), we find

$$z = 4.$$

Substituting the values of y and z in (1), we find

$$x = 2.$$

164. Rule.

1. *Combine one equation with each of the others, eliminating the same unknown quantity, and the number of resulting equations will be one less than the preceding, involving one less unknown quantity.*

2. *Treat the resulting equations in a similar manner, and so on, till one equation is obtained involving one unknown quantity, and deduce the value of this unknown quantity.*

3. *Substitute this value in one of the equations involving two unknown quantities, and deduce the value of a second unknown quantity.*

4. *Substitute these values in one of the equations involving three unknown quantities, and deduce the value of a third unknown quantity, and so on till the values of all the unknown quantities have been determined.*

Scholium.—It sometimes occurs that each equation does not involve all of the unknown quantities. In this case, the process of elimination is facilitated.

165. Examples.

1. Given $\left\{\begin{aligned} x + 2y + 3z &= 26. \\ 2x + y + 5z &= 35. \\ 3x + 6y + z &= 38. \end{aligned}\right\}$ Required x, y, z.

Ans. $x = 3$, $y = 4$, $z = 5$.

2. Given $\left\{\begin{aligned} 3x + 5y &= 35. \\ 5x + 6z &= 43. \\ 6y + 7z &= 45. \end{aligned}\right\}$ Required x, y, z.

Ans. $x = 5$, $y = 4$, $z = 3$.

3. Given $\left\{\begin{aligned} x + y + z &= 53. \\ x + 2y + 3z &= 105. \\ x + 3y + 4z &= 134. \end{aligned}\right\}$ Required x, y, z.

Ans. $x = 24$, $y = 6$, $z = 23$.

4. Given $\left\{\begin{array}{l} x+y+z=29. \\ x+2y+3z=62. \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10. \end{array}\right\}$ Required x, y, z.

Ans. $x=8,\ y=9,\ z=12.$

5. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62. \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47. \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38. \end{array}\right\}$ Required x, y, z.

Ans. $x=24,\ y=60,\ z=120.$

6. Given $\left\{\begin{array}{l} 2x+5y+7z=580. \\ 5x-y+3z=227. \\ 7x+6y-z=173. \end{array}\right\}$ Required x, y, z.

Ans. $x=13,\ y=24,\ z=62.$

7. Given $\left\{\begin{array}{l} x+y+z+u=14. \\ 2x+3y+4z+5u=54. \\ 4x-5y-7z+9u=10. \\ 3x+4y+2z-3u=11. \end{array}\right\}$ Required z, y, z, u.

Ans. $x=2,\ y=3,\ z=4,\ u=5.$

8. Given $\left\{\begin{array}{l} x-3y+5z=10. \\ 2y-4z+6u=16. \\ 3z-5u+7t=24. \\ 4u-6t+8x=-6. \\ 5t-7x+9y=36. \end{array}\right\}$ Required x, y, z, u, t.

Ans. $x=1,\ y=2,\ z=3,\ u=4,\ t=5.$

166. General Formula for Three Unknown Quantities.

$$\text{Given}\quad \left\{\begin{array}{l} ax + dy + gz = p. \\ bx + cy + hz = q. \\ cx + fy + iz = r. \end{array}\right\}\quad \text{Required } x, y, z.$$

Eliminating in the ordinary way, we find

$$x = \frac{fhp + egr + diq - dhr - eip - fgq}{ceg + bdi + afh - aei - bfg - cdh}.$$

Comparing the value of x with the equations, we observe that the first term in the numerator is found by commencing with f, passing up obliquely to the right, taking fhp. The second term is found by commencing with e, taking egr, observing that when we pass out above without crossing three equations, we pass to the foot of the next column to the right. The third term is found by commencing with d, passing to the foot of the next column, and passing up obliquely to the right, taking diq.

The negative terms of the numerator are found by commencing first with d, and passing down obliquely to the right.

Similar laws hold for the denominator which is independent of the second member.

The value of y can be found by a similar process, if we conceive the column containing y to stand at the left.

In like manner the value of z can be found.

If any of the equations do not contain all of the unknown quantities, write the letter in its proper place with the coefficient 0.

If any term is minus, its sign must be considered.

167. Examples.

1. Given $\left\{\begin{aligned} ax + by + cz &= p. \\ dx + ey + fz &= q. \\ gx + hy + iz &= r. \end{aligned}\right\}$ Required x, y, z.

2. Given $\left\{\begin{aligned} 2x + 4y + 3z &= 37. \\ 5x + 6y + 7z &= 74. \\ 10x + 9y + 8z &= 106. \end{aligned}\right\}$ Required x, y, z.

Ans. $x = 3$, $y = 4$, $z = 5$.

3. Given $\left\{\begin{aligned} x - y + z &= 4. \\ x + 2y - 3z &= 4. \\ 2x - y + 2z &= 12. \end{aligned}\right\}$ Required x, y, z.

Ans. $x = 5$, $y = 4$, $z = 3$.

4. Given $\left\{\begin{aligned} ax + by &= c. \\ bx + dz &= e. \\ fy + gz &= h. \end{aligned}\right\}$ Required x, y, z.

SYMMETRICAL EQUATIONS.

168. Definition.

Symmetrical equations are those equations which resolve themselves into each other when certain permutations are made of the letters entering them.

Symmetrical equations admit of elegant solutions. Let them be carefully studied.

169. Illustrations.

1. Given $\left\{\begin{array}{ll}(1) & x+y=a.\\(2) & y+z=b.\\(3) & z+x=c.\end{array}\right\}$ Required x, y, z.

If x be changed to y, y to z, z to x, a to b, b to c, and c to a, (1) will resolve itself into (2), (2) into (3), and (3) into (1).

SOLUTION.

$$(1)+(3)-(2)=(4)\quad 2x=a+c-b.$$

$$\therefore\quad x=\frac{a+c-b}{2}.$$

$$(1)+(2)-(3)=(5)\quad 2y=a+b-c.$$

$$\therefore\quad y=\frac{a+b-c}{2}.$$

$$(2)+(3)-(1)=(6)\quad 2z=b+c-a.$$

$$\therefore\quad z=\frac{b+c-a}{2}.$$

We can obtain the value of y from that of x, and z from that of y, by permuting as above.

2. Given $\left\{\begin{array}{ll}(1) & \dfrac{1}{x}+\dfrac{1}{y}=a.\\(2) & \dfrac{1}{y}+\dfrac{1}{z}=b.\\(3) & \dfrac{1}{x}+\dfrac{1}{z}=c.\end{array}\right\}$ Required x, y, z.

SOLUTION.

$$[(1) + (3) - (2)] \div 2 = (4) \quad \frac{1}{x} = \frac{a + c - b}{2}.$$

Taking the reciprocals, we find

$$x = \frac{2}{a + c - b}.$$

Also, we find

$$y = \frac{2}{b + a - c}.$$

$$z = \frac{2}{c + b - a}.$$

3. Given $\left\{\begin{array}{l} x + y = 7. \\ y + z = 5. \\ x + z = 6. \end{array}\right\}$ Required x, y, z.

Ans. $x = 4$, $y = 3$, $z = 2$.

4. Given $\left\{\begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{8}{15}. \\ \frac{1}{y} + \frac{1}{z} = \frac{12}{35}. \\ \frac{1}{x} + \frac{1}{z} = \frac{10}{21}. \end{array}\right\}$ Required x, y, z.

Ans. $x = 3$, $y = 5$, $z = 7$.

5. Given $\left\{\begin{array}{l} \frac{1}{u} + \frac{1}{x} + \frac{1}{y} = \frac{13}{12}. \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{47}{60}. \\ \frac{1}{y} + \frac{1}{z} + \frac{1}{u} = \frac{19}{20}. \\ \frac{1}{z} + \frac{1}{u} + \frac{1}{x} = \frac{31}{30}. \end{array}\right\}$ Required u, x, y, z.

Ans. $u = 2$, $x = 3$, $y = 4$, $z = 5$.

6. Given $\left\{\begin{array}{l} u+v+w+x+y=15. \\ v+w+x+y+z=20. \\ w+x+y+z+u=19. \\ x+y+z+u+v=18. \\ y+z+u+v+w=17. \\ z+u+v+w+x=16. \end{array}\right\}$ Required u, v, w, x, y, z.

Ans. $u=1$, $v=2$, $w=3$, $x=4$, $y=5$, $z=6$.

170. Problems.

1. What two numbers are those whose sum is 40 and difference 10? *Ans.* 25 and 15.

2. What two numbers are those whose sum is s and difference d? *Ans.* $\frac{1}{2}s+\frac{1}{2}d$ and $\frac{1}{2}s-\frac{1}{2}d$.

3. A person has two horses, and a carriage which is worth \$150. The first horse and carriage are worth twice the second horse, and the second horse and carriage are worth three times the first horse. What is the value of each horse? *Ans.* \$90, \$120.

4. A person has two horses, and a carriage which is worth c dollars. The first horse and the carriage are worth m times the second horse, and the second horse and the carriage are worth n times the first horse. What is the value of each horse? *Ans.* $\frac{(m+1)c}{mn-1}$, $\frac{(n+1)c}{mn-1}$.

5. A and B can do a certain work in 12 days; B can do it alone in 30 days. How long will it take A alone to do it? *Ans.* 20 days.

6. A and B can do a certain work in m days; B can do it alone in n days. How long will it take A alone to do it?

Ans. $\dfrac{mn}{n-m}$ days.

7. A and B can do a certain work in 24 days, A and C in 30 days, B and C in 40 days. How long would it take each to do it?

Ans. A, 40 days; B, 60 days; C, 120 days.

8. A and B can do a certain work in m days, A and C in n days, B and C in p days. How long would it take each to do it?

Ans. A, $\dfrac{2mnp}{np+mp-mn}$; B, $\dfrac{2mnp}{np+mn-mp}$; C, $\dfrac{2mn}{mp+mn-np}$.

9. A has two kinds of money; 10 pieces of the first or 20 pieces of the second make a dollar. How many pieces of each kind must he take in order that 15 pieces may make a dollar?

Ans. 5 of the first, 10 of the second.

10. A has two kinds of money; m pieces of the first or n pieces of the second make a dollar. How many pieces of each kind must he take in order that p pieces may make a dollar?

Ans. $\dfrac{m(p-n)}{m-n}$ of the first, $\dfrac{n(m-p)}{m-n}$ of the second.

11. A's money + 2 times the sum of B's and C's = \$8000, B's + 3 times the sum of A's and C's = \$10500, and C's + 4 times the sum of A's and B's = \$12000. What has each?

Ans. A has \$1000, B \$1500, C \$2000.

12. A's money $+ \, l$ times the sum of B's and C's $= p$ dollars; B's $+ \, m$ times the sum of A's and C's $= q$ dollars; C's $+ \, n$ times the sum of A's and B's $= r$ dollars. How many dollars has each?

$$Ans. \begin{cases} \text{A's} = \dfrac{mnp + lr + lq - lmr - p - nlq}{lm + ln + mn - 2lmn - 1}. \\[2ex] \text{B's} = \dfrac{mnp + lmr + q - mr - mp - lnq}{2mln + 1 - lm - ln - mn}. \\[2ex] \text{C's} = \dfrac{np + lmn + nq - r - mnp - lnq}{lm + mn + ln - 2lmn - 1}. \end{cases}$$

INDETERMINATE EQUATIONS.

171. Definition.

An indeterminate equation is an equation in which the unknown quantities have an infinite number of values.

172. Propositions.

1. *One equation containing two or more unknown quantities is indeterminate.*

Thus, take the equation

$$(1) \quad 2x - 3y = 15.$$

$$\therefore \; (2) \quad x = \frac{15 + 3y}{2}.$$

Let $\qquad y = 1, \quad 2, \quad 3, \quad 4, \quad 5 \ldots.$

Then $\qquad x = 9, \; 10\frac{1}{2}, \; 12, \; 13\frac{1}{2}, \; 15 \ldots.$

Any two corresponding values of x and y will satisfy equation (1).

Again, take the equation

$$(1)\quad 3x + 4y - 5z = 20.$$

$$\therefore\ (2)\quad x = \frac{20 - 4y + 5z}{3}.$$

Now, we can assign values at pleasure to y and z, and deduce the corresponding values of x. Then, the corresponding values of x, y, and z will satisfy equation (1).

2. *Equations are indeterminate when the number of unknown quantities involved exceeds the number of equations.*

For, by eliminating, we can reduce the equations to one equation involving two or more unknown quantities, which is indeterminate, as before shown.

It will not do to supply the lack of equations by deducing other equations from those already given; for the resulting equations would not be independent. Thus, if we have

$$(1)\quad 2x + 3y = 6.$$

and multiply by 2, we have

$$(2)\quad 4x + 6y = 12.$$

Now, in attempting to eliminate, we find that both x and y disappear at the same time.

3. *An equation of the first degree involving but one unknown quantity may be indeterminate in consequence of certain relations existing between the known quantities.*

Thus, take the equation

$$(1)\quad mx + p = nx + q.$$

$$\therefore\ (2)\quad x = \frac{q - p}{m - n}.$$

Now, if $q = p$ and $n = m$, we shall have

$$(3)\qquad x = \frac{0}{0}.$$

This value of x is indeterminate; but the above relations reduce (1) to

$$(4)\quad mx + p = mx + p.$$

But (4) is an *identical equation*, and may be satisfied for any value of x; that is, (4) is an indeterminate equation.

An identical equation affirms no new fact, but merely that a quantity is equal to itself. In this light it is not to be regarded as an equation, in the ordinary sense, expressing a relation between two different objects of thought. We then have *no equation and one unknown quantity;* that is, the number of unknown quantities exceeds the number of equations.

4. *Two equations involving two unknown quantities may be indeterminate in consequence of certain relations existing between the known quantities.*

Thus, take the equations

$$(1)\quad mx + ny = r.$$

$$(2)\quad px + qy = s.$$

$$\therefore\quad (3)\quad x = \frac{qr - ns}{mq - np},$$

$$\text{and}\quad (4)\quad y = \frac{ms - pr}{mq - np}.$$

Now, if we have

$$(5)\qquad qr = ns,$$

$$\text{and}\quad (6)\qquad np = mq,$$

then, by multiplying (5) and (6) together, and reducing, we have

$$(7) \qquad pr = ms.$$

These relations reduce (3) and (4) to

$$x = \frac{0}{0}, \text{ and } y = \frac{0}{0}.$$

But from (5) and (6), we find

$$q = \frac{ns}{r}, \text{ and } p = \frac{mq}{n} = \frac{ms}{r}.$$

These values of p and q reduce (2) to (1), and we then have one equation involving two unknown quantities, which is indeterminate, as has been shown.

All the cases of indetermination, with respect to equations, therefore resolve themselves into this:

Indetermination arises when the number of unknown quantities exceeds the number of equations.

INDETERMINATE PROBLEMS.

173. Definition and Remarks.

An indeterminate problem is a problem which admits of an infinite number of solutions.

When an indeterminate problem is stated, it will be found that the number of unknown quantities exceeds the number of equations.

We may often limit the number of solutions by imposing the condition that the values of the unknown quantities shall be integral numbers.

174. Illustrations.

1. How many sheep at \$4 per head, and cattle at \$25 per head, can be bought for \$1000?

SOLUTION.

Let x = the number of sheep, and y the number of cattle. Then

$$(1)\quad 4x + 25y = 1000,$$

$$\therefore\quad (2)\quad x = 250 - \frac{25}{4}y.$$

Since x and y are integral and positive, y must be a multiple of 4, and $\frac{25}{4}y < 250$.

Let $y =$ 4, 8, 12, 16, 20, 24, 28, 32, 36.

Then $x =$ 225, 200, 175, 150, 125, 100, 75, 50, 25.

2. Find two whole numbers such that 12 times the one minus 13 times the other equals 9.

SOLUTION.

$$(1)\quad 12x - 13y = 9.$$

$$\therefore\quad (2)\quad x = \frac{13y + 9}{12} = y + \frac{y + 9}{12}.$$

Since x and y are whole numbers, $\frac{y + 9}{12}$ must be a whole number. Then let

$$\frac{y + 9}{12} = n.$$

$$\therefore\quad y = 12n - 9.$$

Let $n = 1, 2, 3, 4 \ldots$ Then $\begin{cases} y = 3, 15, 27, 39 \ldots \\ x = 4, 17, 30, 43 \ldots \end{cases}$

3. There are three whole numbers whose sum is 90, and 2 times the first + 3 times the second + 4 times the third is 200. What are those numbers?

SOLUTION.

$$(1) \quad x + y + z = 90.$$

$$(2) \quad 2x + 3y + 4z = 200.$$

$$(2) - (1) \times 2 = (3) \qquad y + 2z = 20.$$

$$\therefore \quad z = 10 - \tfrac{1}{2}y.$$

Let $y = 2, \ 4, \ 6, \ 8, \ 10, \ 12, \ 14, \ 16, \ 18.$

Then $z = 9, \ 8, \ 7, \ 6, \ 5, \ 4, \ 3, \ 2, \ 1,$

and $x = 79, \ 78, \ 77, \ 76, \ 75, \ 74, \ 73, \ 72, \ 71.$

175. Examples.

1. The sum of three whole numbers is 11; and if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products will be 57. What are the numbers?

$$\textit{Ans.} \begin{cases} x = 4, \ 3, \ 2, \ 1. \\ y = 2, \ 4, \ 6, \ 8. \\ z = 5, \ 4, \ 3, \ 2. \end{cases}$$

2. How many calves at $\$3\frac{1}{2}$, sheep at $\$1\frac{1}{2}$, and lambs at $\$\frac{1}{2}$ per head, can be bought for \$100, the number bought being 100?

$$\textit{Ans.} \begin{cases} \text{Calves,} & 1, & 2, & 3. \\ \text{Sheep,} & 47, & 44, & 41. \\ \text{Lambs,} & 52, & 54, & 56. \end{cases}$$

3. What number between 12 and 24, when divided by 2, 3, and 4, will give 1, 2, 3 for the respective remainders?

Ans. 23.

4. Bought three kinds of tea at 6, 9, and 12 shillings per pound. How many pounds of each kind did I buy, provided the entire amount was 40 pounds, and the entire cost £17 8*s*?

$$Ans.\ \begin{cases} 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, \text{ at } 6s. \\ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, \text{ at } 9s. \\ 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, \text{ at } 12s. \end{cases}$$

5. A number is expressed by 3 digits whose sum is 20, and if 16 be subtracted from the number, and the remainder divided by 2, the digits will be inverted. What is the number? *Ans.* 974.

INCOMPATIBLE EQUATIONS.

176. Definition.

Incompatible equations are those equations which can not be satisfied for the same values of the unknown quantities.

177. Propositions.

1. *If the number of independent equations exceeds the number of unknown quantities, these equations will, in general, be incompatible.*

$$\text{Thus, } \begin{cases} (1) & x + y = 8. \\ (2) & x - y = 2. \\ (3) & \dfrac{x}{y} = 2. \end{cases}$$

From (1) and (2) we find $x = 5$ and $y = 3$, which will not satisfy (3); from (2) and (3) we find $x = 4$ and $y = 2$, which will not satisfy (1); from (1) and (3) we find $x = 5\frac{1}{3}$ and $y = 2\frac{2}{3}$, which will not satisfy (2).

2. *If the number of independent equations exceeds the number of unknown quantities, such relations between the known quantities can be found as will make the equations compatible.*

$$\text{Thus,}\quad \begin{cases} (1) & x + y = s. \\ (2) & x - y = d. \\ (3) & \dfrac{x}{y} = q. \end{cases}$$

From (1) and (2) we find

$$x = \tfrac{1}{2}s + \tfrac{1}{2}d.$$

$$y = \tfrac{1}{2}s - \tfrac{1}{2}d.$$

Substituting these values of x and y in (3), we find

$$q = \frac{s + d}{s - d}.$$

This relation of q, s, and d will make the equations compatible. Thus, take the equations

$$(1)\quad x + y = 9,$$

$$(2)\quad x - y = 3,$$

$$(3)\quad \frac{x}{y} = 2,$$

in which this relation exists, since

$$2 = \frac{9 + 3}{9 - 3}.$$

Now, from any two of (1), (2), and (3) we find $x = 6$ and $y = 3$, which satisfy the other, or the equations are compatible.

178. The Problem of the Couriers.

Two couriers were traveling the same road, the one at the rate of a miles per hour, the other at the rate of b miles per hour. At a certain time the distance between them was d miles. When were they together?

Suppose the courier at A travels a miles per hour, and that the courier at B travels b miles per hour. Let the distance AB be denoted by d, and suppose it was noon when the couriers were at A and B respectively. Let t denote the interval of time from noon to the time when they were together. Suppose they traveled toward the right and met at C.

Now, since the rate multiplied by the time equals the distance, we shall have

$$at = \text{AC and } bt = \text{BC}.$$

But $$\text{AC} - \text{BC} = \text{AB}.$$

$$\therefore \quad at - bt = d,$$

or $$(a - b)\, t = d.$$

$$\therefore \quad t = \frac{d}{a - b}.$$

Let us now discuss the value of t on the following suppositions:

$$(1) \quad a > b \text{ and } d > 0.$$

The supposition $a > b$ means that the courier at A travels faster than the courier at B.

The supposition $d > 0$ means that the couriers are not together, and that the distance from A to B is estimated to the right.

In this case, both terms of the fraction $\frac{d}{a-b}$, which expresses the value of t, are positive; hence, t is positive.

The positive value of t indicates that they were together after noon, which corresponds to the assumption that they met at C, at the right of A and B; and this ought to be the case, since at noon the more rapid traveler was following the other, and must overtake him after noon.

$$(2) \quad a < b \text{ and } d > 0.$$

Under this supposition t becomes negative, which indicates that they were together before noon.

This result corresponds to the facts of the ease; for since $b > a$ the courier at B travels faster than the courier at A, and the interval which separates them constantly increases; hence, they can not be together after noon. At noon, the distance between them is d; a little before noon, it must have been less than d; and at a certain time before noon, it must have been 0.

$$(3) \quad a = b \text{ and } d > 0.$$

Under this supposition $t = \frac{d}{a-b} = \frac{d}{0} = \infty$.

This indicates that they will never be together, which result evidently corresponds to the facts of the case; for, since they are separated by a given interval d, and travel at the same rate, they will continually be separated by the interval d, or will never meet.

(4) $a > b$ or $a < b$ and $d = 0$.

Under this snpposition $t = \frac{d}{a-b} = \frac{0}{a-b} = 0$, which indicates that they are together at noon, and were not together again either before or after noon; and this is evidently a correct resnlt; for since $d = 0$, they were together at noon, and since $a > b$ or $a < b$ they travel at different rates, and could not have been together again either before or after noon.

(5) $a = b$ and $d = 0$.

Under this supposition $t = \frac{d}{a-b} = \frac{0}{0}$, which is indeterminate, and indicates that they are together one time as well as another, that they are always together. This resnlt corresponds to the facts of the case; for $d = 0$ indicates that they were together at noon, $a = b$ indicates that they travel at the same rate; hence, they were together both before and after noon, and will continne together so long as they travel according to that law.

INVOLUTION.

POWERS OF MONOMIALS.

179. Definitions.

1. **A power** of a quantity is the product obtained by taking the quantity a certain nnmber of times as a factor.

2. **Involution** is the process of finding the power of a quantity.

3. **The first power** of a quantity is the quantity itself.

4. **The second power, or square,** of a quantity is the product obtained by taking the quantity twice as a factor.

5. **The third power,** or **cube,** of a quantity is the product obtained by taking the quantity three times as a factor.

6. **The n^{th} power** of a quantity is the product obtained by taking the quantity n times as a factor.

180. The Exponent of the Power.

1. The square of the cube or the cube of the square is the 6^{th} power.

Thus,

$$(a^3)^2 = a^3 \times a^3 = a^{3+3} = a^{3\times 2} = a^6,$$

and

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^{2\times 3} = a^6.$$

2. The m^{th} power of the n^{th} power is the mn^{th} power.

Thus,

$$(a^n)^m = a^n \times a^n \times a^n \times \ldots = a^{n+n+n+\ldots \text{ to } m \text{ terms}} = a^{mn}.$$

Conversely,

$$a^{mn} = (a^n)^m.$$

181. The Co-efficient of the Power.

The co-efficient of the n^{th} power is the n^{th} power of the co-efficient. Thus,

$$(2a)^3 = 2a \times 2a \times 2a = 2^3a^3 = 8a^3.$$

In general,

$$(5a^3)^n = 5^n a^{3n}.$$

182. The Sign of the Power.

1. *Any power of a positive quantity is positive.*

This is evident from the fact that when all of the factors are positive, the product is positive.

2. *Any even power of a negative quantity is positive.*

Thus, $(-a)^{2n}$ will represent any even power of any negative quantity; but

$$(-a)^{2n} = [(-a)^2]^n = (a^2)^n = a^{2n}.$$

3. *Any odd power of a negative quantity is negative.*

Thus, $(-a)^{2n+1}$ will represent any odd power of any negative quantity; but

$$(-a)^{2n+1} = (-a)^{2n} \times -a = a^{2n} \times -a = -a^{2n+1}.$$

183. Rule.

Raise the co-efficient to the required power, multiply each exponent by the exponent of the power, and give to the result the sign plus, if the monomial is positive, or if the monomial is negative and the power even; but give to the result the negative sign if the monomial is negative and the power odd.

184. Examples.

1. Square $6a^3b^2$. *Ans.* $36a^6b^4$.
2. Cube $-5a^4b^7$. *Ans.* $-125a^{12}b^{21}$.
3. Raise $-3a^2b^3$ to the 4^{th} power. *Ans.* $81a^8b^{12}$.
4. Raise $8a^4b^3$ to the n^{th} power. *Ans.* $8^na^{4n}b^{3n}$.

5. Raise $-a^2b^m$ to the $2n^{th}$ power. *Ans.* $a^{4n}b^{2mn}$.

6. Raise $-a^2b^2$ to the 10^{th} power. *Ans.* $a^{20}b^{20}$.

7. Raise a^3b^p to the mn^{th} power. *Ans.* $a^{3mn}b^{mnp}$.

8. Raise $2a^mb^n$ to the n^{th} power. *Ans.* $2^na^{mn}b^{n^2}$.

POWERS OF BINOMIALS.

185. Particular Cases.

By actual multiplication, we find that

$$(a+b)^1 = a+b.$$

$$(a+b)^2 = a^2 + 2ab + b^2.$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

186. The Exponents of the Power.

1. The exponent of the leading letter in the first term is equal to the exponent of the power, and diminishes by unity in each succeeding term to the right, till, in the last term, it is 0, or that letter disappears.

2. The exponent of the other letter is 0 in the first term, or that letter does not appear in the first term; and in each succeeding term to the right, the exponent of this letter increases by unity, till, in the last term, it is equal to the exponent of the power.

187. The Co-efficients of the Power.

The co-efficient of the first term is unity, and the co-efficient of any succeeding term is found by multiplying the co-efficient of the term preceding the term required, by the exponent of the leading letter of that term, and dividing the product by the exponent of the other letter plus 1.

188. Demonstration.

We shall prove that if the above laws hold for the n^{th} power, they will hold for the $(n+1)^{th}$ power.

If these laws hold for the n^{th} power, we shall have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \ldots b^n.$$

We shall find the $(n+1)^{th}$ power by multiplying the n^{th} power by $a+b$, thus:

$$\begin{array}{l}
a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \ldots + b^n. \\
a+b \\
\hline
a^{n+1} + na^nb + \frac{n(n-1)}{1\,.\,2}a^{n-1}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-2}b^3 + \ldots + ab^n. \\
\quad + a^nb + \qquad na^{n-1}b^2 + \qquad \frac{n(n-1)}{1\,.\,2}a^{n-2}b^3 + \ldots + b^{n+1}. \\
\hline
a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1\,.\,2}a^{n-1}b^2 \\
\qquad + \frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}a^{n-2}b^3 + \ldots + b^{n+1}.
\end{array}$$

$$\text{The } (r+1)^{th} \text{ co-efficient} = \frac{n(n-1)\,..\,(n-r+1)}{1\,.\,2 \quad ..\quad r}$$

$$+\frac{n(n-1)..(n-r+2)}{1\,.\,2\,..\,r-1} = \frac{(n+1)n(n-1)\,..\,(n-r+2)}{1\,.\,2\,.\,3 \quad ..\quad r}.$$

Hence, if the laws hold for the n^{th} power, they hold for the $(n+1)^{th}$ power; that is, if the laws hold for any power, they hold for the next higher power; but, by actual trial, they are found to hold for all the powers up to the 5^{th}; and if they hold for the 5^{th}, then they hold for the 6^{th}, and hence for the 7^{th}, and so on to any extent.

189. The Signs of the Power.

1. If both terms of the binomial are positive, the signs of all the terms of the power are positive.

2. If both terms are negative, the signs of all the terms of any even power are positive, and of any odd power negative.

3. If one term is positive and the other negative, the signs of all the terms containing the odd powers of the negative term will be negative, the signs of the other terms will be positive.

190. The Binomial Formula.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \ldots + b^n.$$

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \ldots + b^n.$$

191. Application Illustrated.

1. $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

2. $(2a^2+3b)^3 = (2a^2)^3 - 3(2a^2)^2(3b) + 3(2a^2)(3b)^2 + (3b)^3 = 8a^6 + 36a^4b + 54a^2b^2 + 27b^3$

3. $(a+b+c)^3 = [(a+b)+c]^3 = (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

192. Examples.

1. $(a+b)^4$.
2. $(a-b)^5$.
3. $(x+y)^7$.
4. $(a+b)^8$.
5. $(x-y)^{10}$.
6. $(x+y)^{12}$.
7. $(3a+2b^2)^3$.
8. $(5a-2b)^4$.
9. $(2ab-3ac)^4$.
10. $(a^m+b^n)^5$.
11. $(a+b-c)^4$.
12. $(a-b+c)^5$.

EVOLUTION.

SQUARE ROOT OF NUMBERS.

193. Definition.

The square root of a number is a number which multiplied by itself will produce the given number.

194. Pointing into Periods.

The square of the units of a number will produce one or two figures, and hence will fall in the first period of two figures at the left of the decimal point.

The square of the tens will fall in the second period, the square of the hundreds in the third period, etc.

The square of the tenths will fall in the first period at the right of the point, the square of the hundredths in the second period, etc.

Hence, in preparing a number for the purpose of extracting its root, we point it into periods of two places each, commencing at units, by placing a dot over the right-hand figure of each period. Thus,

$$3\dot{4}7\dot{6}.8\dot{9}7\dot{9} \text{ and } \dot{7}8\dot{4}.6\dot{3}7\dot{0}.$$

195. Formulas for Extracting the Square Root.

The formula $(a+b)^2 = a^2 + 2ab + b^2$ may take the form

$$(1) \quad (a+b)^2 = a^2 + (2a+b)\,b.$$

The square of the sum of two quantities is equal to the square of the first, plus twice the first plus the second into the second.

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + [2(a+b)+c]\,c.$$

Omitting $[(a+b)+c]^2$, and substituting the value of $(a+b)^2$ as found in (1), we have

$$(2) \quad (a+b+c)^2 = a^2 + (2a+b)\,b + [2(a+b)+c]\,c.$$

By a similar process, we find

$$(3) \quad (a+b+c+d)^2 = a^2 + (2a+b)\,b + [2(a+b)+c]\,c + [2(a+b+c)+d]\,d.$$

In general, the square of the sum of any number of terms is equal to the square of the first term, plus twice the first plus the second into the second, plus twice the sum of the first and second plus the third into the third, plus twice the sum of the first, second, and third plus the fourth into the fourth, plus, etc.

196. Application.

1. Let us find the square root of 4096.

OPERATION.

$$\begin{array}{rl} a^2 + (2a+b)\,b = & \dot{4}0\dot{9}6 \,\lfloor\, \overset{a\,b}{64} \\ a^2 = & \underline{36} \\ 2a+b = 124\,) & 496 = (2a+b)\,b \\ & \underline{496} \end{array}$$

We find $a = 6$ tens, the first term of the root. Subtracting a^2 from the first period and bringing down the next period, we have $496 = (2a + b)\,b$, the first dividend. The trial divisor $2a = 120$. Dividing 496 by 120, we obtain $4 = b$, and a remainder, as there ought to be, since the trial divisor is too small. The complete divisor $2a + b = 124$, which is contained in 496 just 4 times. Hence, 64 is the square root of 4096.

2. Let us find the square root of 119025.

OPERATION.

$$\begin{array}{rl} a^2 + (2a+b)b + [2(a+b)+c]c = & 1\dot{1}9\dot{0}2\dot{5} \,\lfloor\, \overset{a\,b\,c}{345} \\ a^2 = & \underline{9} \\ 2a+b = 64\,) & 290 \quad = (2a+b)\,b + \\ & \underline{256} \quad = (2a+b)\,b \\ 2\,(a+b)+c = 685\,) & 3425 = [2\,(a+b)+c]\,c \\ & \underline{3425} \end{array}$$

197. Rule.

1. *Point off the expression into periods of two places each, commencing with units.*

2. *Find the root of the greatest square in the left-hand period, place this root at the right, subtract its square from the left period, and to the remainder bring down the next period.*

3. *Take twice the root already found reduced to the denomination of the next term of the root for a trial divisor, and find the next term of the root; the trial divisor plus the second term of the root will be the true divisor; then divide and proceed as before.*

Sch. 1. If the number contains decimal places, there will be as many decimal places in the root as there are decimal periods.

Sch. 2. If the number is not a perfect square, there will be a remainder, to which annex a period of ciphers, and proceed as before.

Sch. 3. To extract the square root of a common fraction, extract the square root of both terms, or first reduce it to a decimal.

Sch. 4. After the process is thoroughly understood, the letters can be omitted.

198. Examples.

Extract the square root of the following:

1. 625. *Ans.* 25.
2. 15625. *Ans.* 125.
3. 85264. *Ans.* 292.

4. 547.56. *Ans.* 23.4.

5. 5499025. *Ans.* 2345.

6. $\frac{9}{16}$. *Ans.* $\frac{3}{4}$.

7. 61.1524. *Ans.* 7.82.

8. $\frac{3}{4}$. *Ans.* .866 +.

9. 342694144. *Ans.* 18512.

10. 1143881586576. *Ans.* 1069524.

SQUARE ROOT OF MONOMIALS.

199. Illustration.

Since the square of a monomial is obtained by squaring its co-efficient, and multiplying each exponent by 2, and since the square is plus, whether the monomial is plus or minus, it follows that the square root of a monomial is obtained by extracting the square root of the co-efficient, dividing each exponent by 2, and giving to the result the double sign (±), plus or minus. Thus,

$$\sqrt{36a^4b^2} = \pm 6a^2b.$$

200. Rule.

Extract the square root of the co-efficient, divide each exponent by 2, *retain all the letters, and give to the result the double sign* ±.

201. Examples.

1. Find the square root of $16a^2b^4c^2$.

2. Find the square root of $25a^2b^8c^{10}$.

3. Find the square root of $64a^6b^2c^8$.

4. Find the square root of $81a^{2m}b^{4n}$

SQUARE ROOT OF POLYNOMIALS.

202. Illustration.

$$(a+b+c)^2 = a^2 + (2a+b)\,b + [2\,(a+b)+c]\,c.$$

Let us now find the square root of the second member developed.

OPERATION.

$$\begin{array}{rl}
 & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \;(a+b+c. \\
 & \underline{a^2} \\
2a+b) & 2ab + b^2 \\
 & \underline{2ab + b^2} \\
2a+2b+c) & 2ac + 2bc + c^2 \\
 & \underline{2ac + 2bc + c^2}
\end{array}$$

203. Rule.

1. *Arrange the polynomial with reference to a certain letter.*

2. *Extract the square root of its first term for the first term of the root.*

3. *Subtract the square of the first term of the root from the given polynomial, and bring down two terms of the polynomial for the first dividend.*

4. *Take twice the first term of the root for a trial divisor, by which divide the first term of the dividend. and the quotient will be the second term of the root.*

5. *To the trial divisor add the second term of the root, and the sum will be the complete divisor.*

6. *Multiply the complete divisor by the second term of the root, subtract the product from the dividend, bring down additional terms of the dividend, and proceed as before.*

204. Examples.

Find the square root of the following:

1. $4x^4 - 12x^3 + 5x^2 + 6x + 1.$
   Ans. $2x^2 - 3x - 1.$

2. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6.$
   Ans. $x^3 - 3ax^2 + 3a^2x - a^3.$

3. $a^6 + 4a^5 + 10a^4 + 20a^3 + 25a^2 + 24a + 16.$
   Ans. $a^3 + 2a^2 + 3a + 4.$

4. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
   Ans. $a^2 + 2ab + b^2.$

5. $4a^4 + 12a^3b + 13a^2b^2 + 6ab^3 + b^4$
   Ans. $2a^2 + 3ab + b^2.$

CUBE ROOT OF NUMBERS.

205. Pointing into Periods.

Since the cube of units falls in the first period of three figures at the left of the decimal point, the cube of tens in the second, etc., the number is pointed into periods of three places each, commencing at the decimal point. Thus,

$$27\dot{6}34\dot{0}78\dot{9} \text{ and } 7\dot{8}67\dot{4}.67\dot{5}83\dot{2}.$$

206. Formulas for Extracting the Cube Root.

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

or, (1) $(a + b)^3 = a^3 + (3a^2 + 3ab + b^2)b.$

Hence, *The cube of the sum of two terms is equal to the cube of the first, plus the sum of three times the square of the first, three times the first by the second, and the square of the second by the second.*

Applying this principle, we have

$$[(a+b)+c]^3 = (a+b)^3 + [3(a+b)^2 + 3(a+b)c + c^2]c.$$

Omitting the parenthesis in the first member, and substituting the value of $(a+b)^3$ in the second member as found in (1), we have

$$\text{(2)}\quad (a+b+c)^3 = a^3 + (3a^2+3ab+b^2)b \\ + [3(a+b)^2 + 3(a+b)c + c^2]c.$$

By a similar process, we find

$$\text{(3)}\quad (a+b+c+d)^3 = a^3 + (3a^2+3ab+b^2)b \\ + [3(a+b)^2 + 3(a+b)c + c^2]c \\ + [3(a+b+c)^2 + 3(a+b+c)d + d^2]d.$$

The formula expressed in words gives the law,

The cube of the sum of any number of terms is equal to the cube of the first, plus the sum of three times the square of the first, three times the first by the second and the square of the second by the second; plus the sum of three times the square of the sum of the first and second, three times the sum of the first and second by the third and the square of the third by the third; plus the sum of three times the square of the sum of the first, second, and third, three times the sum of the first, second, and third by the fourth and the square of the fourth by the fourth, plus, etc.

207. Application.

1. Let us find the cube root of 91125.

OPERATION.

$$\begin{array}{rrr|l}
a^3 + (3a^2 + 3ab + b^2)\,b & & = 9\dot{1}12\dot{5} & (\overset{a}{4}\overset{b}{5} \\
 & a^3 & = 64 & \\
3a^2 = 3 \times 40^2 & & = 4800 & 27125 \\
3ab = 3 \times 40 \times 5 & & = 600 & 27125 \\
b^2 = & 5^2 & = 25 & 0 \\
3a^2 + 3ab + b^2 & & = 5425 &
\end{array}$$

2. Let us find the cube root of 94818816.

OPERATION.

$$
\begin{array}{rrr|l}
 & & & 9\dot{4}81\dot{8}81\dot{6}\,(\overset{a}{4}\overset{b}{5}\overset{c}{6} \\
 & a^3 = & & 64 \\
3a^2 = & 3 \times 40^2 = & 4800 & 30818 \\
3ab = & 3 \times 40 \times 5 = & 600 & 27125 \\
b^2 = & 5^2 = & 25 & 3693816 \\
 & & 5425 & 3693816 \\
3\,(a+b)^2 = & 3 \times 450^2 = & 607500 & \\
3\,(a+b)\,c = & 3 \times 450 \times 6 = & 8100 & \\
c^2 = & 6^2 = & 36 & \\
\multicolumn{2}{r}{3\,(a+b)^2 + 3\,(a+b)\,c + c^2 =} & 615636 &
\end{array}
$$

208. Rule.

1. *Point off the expression into periods of three places each, counting from the decimal point.*

2. *Find the root of the greatest cube in the left period, which place at the right, subtract its cube from the left period, and to the remainder bring down the next period.*

3. *Reduce the term of the root already found to the denomination of the next term of the root, and take three times its square for a trial divisor, from which find the second term of the root.*

4. *To the trial divisor add three times the first term of the root by the second, also the square of the second, and the sum will be the true divisor. Divide, and to the remainder bring down the next period, and continue this process till all of the periods have been brought down.*

Sch. 1. If the number is wholly or partly decimal, there will be as many decimal places in the root as there are decimal periods used.

Sch. 2. If the number is not a perfect cube, the process can be continued by bringing down decimal periods of ciphers.

Sch. 3. Since the trial divisor is too small, it may give a quotient too large; if this is the case, it will appear when the true divisor is found.

Sch. 4. The cube root of a common fraction can be found by extracting the root of both terms, or by first reducing it to a decimal, and then extracting the root.

Sch. 5. If the divisor is not contained in the dividend, place a cipher in the root and bring down the next period; and, in finding the next trial divisor, reduce the root already found to the denomination of the next term of the root.

Sch. 6. We shall now give a contracted method of finding the trial divisor. Take the general formula

$$\begin{aligned}(a+b+c+d+\dots)^3 = a^3 &+ (3a^2+3ab+b^2)\,b\\ &+ [3\,(a+b)^2 + 3\,(a+b)\,c + c^2]\,c,\\ &+ [3\,(a+b+c)^2 + 3\,(a+b+c)\,d + d^2]\,d + \dots\end{aligned}$$

We find that

$$\begin{aligned}3\,(a+b)^2 &= 3a^2 + 6ab + 3b^2,\\ 3\,(a+b+c)^2 &= 3\,(a+b)^2 + 6\,(a+b)\,c + 3c^2,\\ 3\,(a+b+c+d)^2 &= 3\,(a+b+c)^2 + 6\,(a+b+c)\,d + 3d^2.\end{aligned}$$

That is, *Each trial divisor* = *the sum of the first term of the next preceding true divisor, twice its second term, and three times its third term.*

Two ciphers are annexed to reduce it to the required order.

209. Examples.

1. What is the cube root of 13824? *Ans.* 24.
2. What is the cube root of 54872? *Ans.* 38.
3. What is the cube root of 117649? *Ans.* 49.
4. What is the cube root of 941192? *Ans.* 98.
5. What is the cube root of 3048625? *Ans.* 145.
6. What is the cube root of 102503232? *Ans.* 468.
7. What is the cube root of 341532099? *Ans.* 699.
8. What is the cube root of 491169069? *Ans.* 789.
9. What is the cube root of 158252.632929? *Ans.* 54.09.
10. What is the cube root of 491916472984? *Ans.* 7894.

THE CUBE ROOT OF MONOMIALS.

210. Rule.

Extract the cube root of the co-efficient, divide each exponent by 3, retain all the letters, and give to the result the same sign as that of the given monomial.

211. Examples.

1. Find the cube root of $8a^3b^6c^9$. *Ans.* $2ab^2c^3$.
2. Find the cube root of $-27a^3b^3c^{12}$. *Ans.* $-3abc^4$.
3. Find the cube root of $64a^9b^{12}c^{15}$. *Ans.* $4a^3b^4c^5$.
4. Find the cube root of $-125a^{3m}b^{6n}$. *Ans.* $-5a^mb^{2n}$.
5. Find the cube root of $216a^{6p}b^{9q}c^{12r}$. *Ans.* $6a^{2p}b^{3q}c^{4r}$.

THE CUBE ROOT OF POLYNOMIALS.

212. Illustration.

We have already found that

$$(a+b+c)^3 = a^3 + (3a^2+3ab+b^2)\,b \\ + [3\,(a+b)^2 + 3\,(a+b)\,c + c^2]\,c.$$

Let us now find the cube root of the second member.

OPERATION.

$$\begin{array}{l|l} \multicolumn{1}{l}{} & a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c \;\underline{\lfloor a+b+c} \\ \multicolumn{1}{l}{} & \underline{a^3} \\ 3a^2+3ab+b^2 & (3a^2+3ab+b^2)b \\ & \underline{(3a^2+3ab+b^2)b} \\ 3(a+b)^2+3(a+b)c+c^2 & [3(a+b)^2+3(a+b)c+c^2]c \\ & \underline{[3(a+b)^2+3(a+b)c+c^2]c} \end{array}$$

213. Rule.

1. *Arrange the polynomial with reference to a certain letter, extract the cube root of its first term for the first term of the root.*

2. *Subtract its cube from the polynomial, bring down three terms of the polynomial for the first dividend, and divide the first term of the dividend by the trial divisor, which is three times the square of the first term of the root, and the quotient will be the second term of the root.*

3. *To the trial divisor add three times the first term of the root by the second, also the square of the second, which will give the true divisor.*

4. *Multiply the true divisor by the second term of the root, subtract the product from the first dividend, and bring down additional terms of the polynomial, and the result will be the second dividend.*

5. *Divide the first term of the second dividend by the trial divisor, which is three times the square of the first term of the root, and the quotient will be the third term of the root.*

6. *Then find the true divisor by taking three times the square of the sum of the first and second terms of the root, three times the sum of the first and second terms by the third, also the square of the third.*

7. *Multiply the true divisor by the third term of the root, subtract the product from the dividend, and proceed in like manner till the required root is obtained.*

214. Examples.

1. $a^3 + 3a^2b + 3ab^2 + b^3$. *Ans.* $a + b$.

2. $x^6 + 6x^5 - 40x^3 + 96x - 64$. *Ans.* $x^2 + 2x - 4$.

3. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
Ans. $x^2 - 2x + 1$.

4. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
Ans. $a^2 + 2ab + b^2$.

5. $8x^6 - 6bax^5 + 66a^2x^4 - 63a^3x^3 + 33a^4x^2 - 9a^5x + a^6$.
Ans. $2x^2 - 3ax + a^2$.

THE HIGHER ROOTS OF NUMBERS.

215. When the Index of the Root is Composite.

Let mn denote the index; then we can apply the principle: The mn^{th} root of a quantity is equal to the m^{th} root of the n^{th} root of that quantity. That is,

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

For, let

$$(1) \quad \sqrt[m]{\sqrt[n]{a}} = p.$$

$$(1)^m = (2) \quad \sqrt[n]{a} = p^m.$$

$$(2)^n = (3) \quad a = p^{mn}.$$

$$\sqrt[mn]{(3)} = (4) \quad \sqrt[mn]{a} = p.$$

$$\therefore \quad \sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

Thus,

$$\sqrt[4]{a} = \sqrt{\sqrt{a}}, \ \sqrt[6]{a} = \sqrt[3]{\sqrt{a}}, \ \sqrt[8]{a} = \sqrt{\sqrt{\sqrt{a}}}, \text{ etc.}$$

216. Examples.

1. Find the value of $\sqrt[4]{20736}$. *Ans.* 12.

2. Find the value of $\sqrt[8]{1679616}$. *Ans.* 6.

3. Find the value of $\sqrt[6]{262144}$. *Ans.* 8.

4. Find the value of $\sqrt[9]{387420489}$. *Ans.* 9.

5. Find the value of $\sqrt[12]{64339296875 \times 52521875}$. *Ans.* 35.

217. To extract Roots when the Index is Prime.

Let n denote the index of the root; then, by reversing the Binomial Formula, Art. 190, we have the rule:

218. Rule.

1. *Point off the expression into periods of* n *places each, counting from the decimal point.*

2. *Find the greatest* n^{th} *power in the first period, place its* n^{th} *root at the right for the first term of the root, subtract the* n^{th} *power of this term of the root from the first period, and to the remainder bring down the next period for the first dividend.*

3. *Reduce the first term of the root to the next lower denomination, and take* n *times its* $(n-1)^{th}$ *power for a trial divisor, from which find the second term of the root.*

4. *Find the* n^{th} *power of the root already found, subtract the result from the first two periods, to the remainder bring down the next period for a second dividend, and proceed as before.*

219. Examples.

1. Find the value of $\sqrt[5]{33554432}$. *Ans.* 32.

2. Find the value of $\sqrt[7]{6103515625}$. *Ans.* 25.

3. Find the value of $\sqrt[5]{36936242722357}$. *Ans.* 517.

4. Find the value of $\sqrt[7]{64339296875}$. *Ans.* 35.

5. Find the value of $\sqrt[11]{36028797018963968}$. *Ans.* 32.

THE HIGHER ROOTS OF MONOMIALS.

220. Principles.

1. *Every even power of a quantity is positive.*

2. *Every odd power of a quantity has the same sign as the quantity itself.* Hence,

1. *An even root of a positive quantity is positive or negative.*

2. *An even root of a negative quantity is impossible.* Thus, $\sqrt{-a}$, $\sqrt[4]{-a}$, $\sqrt[2n]{-a}$ are called *imaginary* quantities.

3. *An odd root of a quantity has the same sign as the quantity itself.*

Let n denote the index of the root; then we have the rule:

221. Rule.

Extract the n^{th} *root of the co-efficient, divide each exponent by* n, *retain all the letters, give to the result the double sign* $\pm$ *if* n *is even, but the same sign as the monomial if* n *is odd.*

222. Examples.

1. Find the 5^{th} root of $32a^{10}b^5$. *Ans.* $2a^2b$.
2. Find the 6^{th} root of $64a^{12}b^{18}$. *Ans.* $\pm 2a^2b^3$.
3. Find the 7^{th} root of $-78125a^7b^{14}$. *Ans.* $-5ab^2$.
4. Find the n^{th} root of $a^{2n}b^{mn}$. *Ans.* $\pm a^2b^m$.
5. Find the $2n$ root of $-a^{4n}$. *Ans.* $\sqrt[2n]{-a^{4n}}$.

THE HIGHER ROOTS OF POLYNOMIALS.

223. Rule.

1. *Arrange the polynomial according to the ascending or descending powers of a certain letter.*

2. *Extract the* n^{th} *root of the first term of the polynomial, place the result at the right for the first term of the root, and subtract the* n^{th} *power of this term of the root from the given polynomial.*

3. *Divide the first term of the remainder by* n *times the* $(n-1)^{th}$ *power of the first term of the root, and the quotient will be the second term of the root.*

4. *Raise the root already found to the* n^{th} *power, subtract the result from the given polynomial, and proceed as before.*

224. Examples.

1. Find the 4^{th} root of $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
Ans. $x - y$.

2. Find the 5^{th} root of $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
Ans. $a + b$.

3. Find the 4^{th} root of $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.
Ans. $2a - 3b$.

4. Find the 6^{th} root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.
Ans. $a - 1$.

RADICALS.

225. Definitions.

1. A radical quantity is an indicated root of an imperfect power. Thus, $\sqrt{5}$, $\sqrt[3]{7}$, $8^{\frac{1}{4}}$, etc., are radicals. Radicals are also called *irrational quantities*, or *surds*.

2. A rational quantity is one which can be expressed without the radical sign or fractional exponent. Thus, 5, $\sqrt{9}$, etc. Rational quantities can always be expressed under the radical form. Thus, $5 = \sqrt{25}$, $3 = \sqrt[3]{27}$, etc. By the definition of a root, it follows that $\sqrt{a}$ squared $= a$, $\sqrt[3]{a}$ cubed $= a$, $\sqrt[n]{a}$ raised to the n^{th} power $= a$; also, the n^{th} root of $a^n = a$, the n^{th} root of $(\sqrt[n]{a})^n = \sqrt[n]{a}$, etc. Radicals are of different degrees, as the $2d$, $3d$, $\ldots n^{th}$.

3. **The index**, denoting the degree of the radical, is the figure or letter placed in the angle of the radical sign, or the denominator of the fractional exponent. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[n]{a}$, or $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{n}}$, denote respectively the square root of a, the cube root of a, the n^{th} root of a. A radical of the second degree is often called *a quadratic surd*; one of the third degree, *a cubic surd.*

4. **The co-efficient** of a radical is a factor which is to be multiplied into the radical. Thus, $4\sqrt[3]{a}$; 4 is the co-efficient of $\sqrt[3]{a}$, and 1 understood is the co-efficient of $\sqrt{a}$.

5. **Similar radicals** are those which have a common index and the same quantity under the radical sign. Thus, $a\sqrt[n]{b}$, and $-c\sqrt[n]{b}$.

REDUCTION OF RADICALS.

226. Case I.

To reduce radicals to their most simple form.

This reduction depends upon the principle: *The* nth *root of the product of two quantities is equal to the product of the* nth *roots of those quantities, and conversely.* For,

$$(1) \quad (\sqrt[n]{ab})^n = ab.$$

$$(2) \quad (\sqrt[n]{a} \times \sqrt[n]{b})^n = ab.$$

$$\therefore \quad (3) \quad (\sqrt[n]{ab})^n = (\sqrt[n]{a} \times \sqrt[n]{b})^n.$$

$$\sqrt[n]{(3)} = (4) \quad \sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

Thus, $\sqrt{12a^3} = \sqrt{4a^2} \times \sqrt{3a} = 2a\sqrt{3a}.$

Also $2\sqrt[3]{54a^8} = 2\sqrt[3]{27a^6} \times \sqrt[3]{2a^2} = 6a^2\sqrt[3]{2a^2}.$

227. Rule.

Resolve the given radical into two factors, one of which is the greatest factor which is a perfect power of the degree indicated, extract its root, which multiply by the co-efficient, and place the result as the co-efficient of the radical factor.

228. Examples.

1. Simplify $\sqrt{8a^3b^2c}$. *Ans.* $2ab\sqrt{2ac}$.
2. Simplify $\sqrt{50a^4b^8c^2}$. *Ans.* $5a^2b^4c\sqrt{2}$.
3. Simplify $\sqrt{192a^5b^6c^7}$. *Ans.* $8a^2b^3c^3\sqrt{3ac}$.
4. Simplify $\sqrt[3]{16a^3b^4}$. *Ans.* $2ab\sqrt[3]{2b}$.
5. Simplify $\sqrt[3]{500a^7b^9c^{11}}$. *Ans.* $5a^2b^3c^3\sqrt[3]{4ac^2}$.
6. Simplify $\sqrt[4]{243a^4b^8c^5}$. *Ans.* $3ab^2c\sqrt[4]{3c}$.
7. Simplify $\sqrt[4]{625a^4b^4c}$. *Ans.* $5ab\sqrt[4]{c}$.
8. Simplify $3\sqrt[5]{64a^5b^{10}c^{12}}$. *Ans.* $6ab^2c^2\sqrt[5]{2c^2}$.
9. Simplify $5\sqrt[5]{486a^{5m}b^{10n}c}$. *Ans.* $15a^mb^n\sqrt[5]{2c}$.
10. Simplify $7\sqrt[n]{a^{2n}b^{mn}c^p}$. *Ans.* $7a^2b^m\sqrt[n]{c^p}$.

229. Case II.

To pass the co-efficient under the radical sign.

Since $a = \sqrt[n]{a^n}$, $a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{a^nb}$.

230. Rule.

Raise the co-efficient to the nth *power, and multiply it into the quantity under the radical sign.*

231. Examples.

1. $5\sqrt{2}$. *Ans.* $\sqrt{50}$.

2. $a\sqrt[5]{b}$. *Ans.* $\sqrt[5]{a^5b}$.

3. $3a\sqrt[n]{c}$. *Ans.* $\sqrt[n]{3^na^nc}$.

4. $a^2\sqrt{b}$. *Ans.* $\sqrt{a^4b}$.

5. $3\sqrt[3]{3}$. *Ans.* $\sqrt[3]{81}$.

6. $p\sqrt[n]{q}$. *Ans.* $\sqrt[n]{p^nq}$.

7. $(a-b)\sqrt{a-b}$. *Ans.* $\sqrt{(a-b)^3}$.

8. $\sqrt{a}\sqrt[4]{b}$. *Ans.* $\sqrt[4]{a^2b}$.

9. $\sqrt[n]{a}\sqrt[2n]{b}$. *Ans.* $\sqrt[2n]{a^2b}$.

10. $\sqrt[n]{a}\sqrt[mn]{b}$. *Ans.* $\sqrt[mn]{a^mb}$.

232. Case III.

To reduce radicals to equivalent radicals having a common index.

$$(1) \quad (\sqrt[n]{a})^n = a.$$

$$(2) \quad (\sqrt[mn]{a})^{mn} = a.$$

$$\therefore \quad (3) \quad (\sqrt[n]{a})^n = (\sqrt[mn]{a})^{mn}.$$

$$\sqrt[n]{(3)} = (4) \quad \sqrt[n]{a} = (\sqrt[mn]{a})^m.$$

But $$(\sqrt[mn]{a})^m = \sqrt[mn]{a} \times \sqrt[mn]{a} \times \ldots = \sqrt[mn]{a^m}.$$

$$\therefore \quad \sqrt[n]{a} = \sqrt[mn]{a^m}.$$

Hence, we may multiply the index of the radical by m, *if we raise the quantity under the radical sign to the* mth *power; and conversely, we can divide the index by* m, *if we extract the* mth *root of the quantity under the radical sign.*

This principle enables us to reduce radicals to a common index, for which we have the rule:

233. Rule.

Multiply each index by the number which will give for the product the least common multiple of the indices, and raise the quantity under each radical sign to the power denoted by the corresponding multiplier.

234. Examples.

1. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$ to a common index.

Ans. $\sqrt[12]{64}$, $\sqrt[12]{81}$, $\sqrt[12]{125}$.

2. Reduce $\sqrt[3]{a}$, $\sqrt[6]{b}$, $\sqrt{c}$ to a common index.

Ans. $\sqrt[6]{a^2}$, $\sqrt[6]{b}$, $\sqrt[6]{c^3}$.

3. Reduce $\sqrt{a^3}$, $\sqrt[3]{a^2}$, $\sqrt[6]{a^3}$ to a common index.

Ans. $\sqrt[6]{a^9}$, $\sqrt[6]{a^4}$, $\sqrt[6]{a^3}$.

4. Reduce $\sqrt{2}$, $\sqrt[4]{3}$, $\sqrt[6]{5}$, $\sqrt[8]{7}$ to a common index.

Ans. $\sqrt[24]{4096}$, $\sqrt[24]{729}$, $\sqrt[24]{625}$, $\sqrt[24]{343}$.

5. Reduce $\sqrt[m]{a}$, $\sqrt[n]{b}$ to a common index.

Ans. $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

ADDITION AND SUBTRACTION OF RADICALS.

235. Rule.

Make the radicals similar, if possible; take the sum or difference of their co-efficients, and annex the common radical; otherwise, connect them by the sign plus or minus.

236. Examples.

1. Find the sum of $3\sqrt{5}$ and $7\sqrt{5}$. *Ans.* $10\sqrt{5}$.

2. Find the sum of $3\sqrt{5}$ and $\sqrt{20}$. *Ans.* $5\sqrt{5}$.

3. Find the sum of $7\sqrt{10}$ and $2\sqrt{90}$. *Ans.* $13\sqrt{10}$.

4. Find the sum of $4\sqrt{20}$ and $3\sqrt{45}$. *Ans.* $17\sqrt{5}$.

5. Find the sum of $a\sqrt{b}$ and $c\sqrt{b}$. *Ans.* $(a+c)\sqrt{b}$.

6. Find the sum of $3\sqrt{a^2b}$ and $2c\sqrt{4b}$. *Ans.* $(3a+4c)\sqrt{b}$.

7. Find the sum of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$. *Ans.* $\frac{14}{45}\sqrt{15}$.

8. Find the difference of $7\sqrt{12}$ and $4\sqrt{3}$. *Ans.* $10\sqrt{3}$.

9. Find the difference of $8\sqrt{32}$ and $7\sqrt{18}$. *Ans.* $11\sqrt{2}$.

10. Find the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$. *Ans.* $\frac{1}{30}\sqrt{6}$.

11. Find the difference of $4a\sqrt[6]{9a^4}$ and $\sqrt[3]{24a^5}$.

Ans. $2a\sqrt[3]{3a^2}$.

12. Find the difference of $\sqrt[3]{40a^6b}$ and $\sqrt[3]{5a^5b}$.

Ans. $a\sqrt[3]{5a^2b}$.

13. Find the difference of $\sqrt[4]{16a^4b^2}$ and $\sqrt{4c^2b}$.

Ans. $2(a-c)\sqrt{b}$.

14. Find the difference of $3a\sqrt[5]{b}$ and $\sqrt[10]{a^{10}b^2}$.

Ans. $2a\sqrt[5]{b}$.

15. Find the difference of $6\sqrt[4]{a^2}$ and $3\sqrt[6]{a^3}$.

Ans. $3\sqrt{a}$.

MULTIPLICATION AND DIVISION OF RADICALS.

237. Rule.

Reduce the radicals to a common index, take the product or quotient of the co-efficients for the co-efficient, and the product or quotient of the quantities under the radical sign for the radical part.

238. Examples.

1. Multiply $\sqrt{a}$ by $\sqrt[3]{c}$. *Ans.* $\sqrt[6]{a^3c^2}$.

2. Multiply $\sqrt[m]{a}$ by $\sqrt[n]{b}$. *Ans.* $\sqrt[mn]{a^nb^m}$.

3. Multiply $5\sqrt{2a}$ by $6\sqrt{2a}$. *Ans.* $60a$.

4. Multiply $10a\sqrt{b}$ by $3b\sqrt[3]{a}$. *Ans.* $30ab\sqrt[6]{a^2b^3}$.

5. Multiply $5\sqrt{18}$ by $6\sqrt{2}$. *Ans.* 180.

6. Multiply $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c}$, $\sqrt[6]{d}$ *Ans.* $\sqrt[12]{a^6b^4c^3d^2}$.

7. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$. *Ans.* $\sqrt[4]{2}$.

8. Multiply $\sqrt[3]{\frac{1}{2}}$ by $\sqrt[14]{6}$. *Ans.* $\sqrt[42]{\frac{27}{2048}}$.

9. Multiply $\sqrt[m]{a+b}$ by $\sqrt[n]{a+b}$. *Ans.* $\sqrt[mn]{(a+b)^{m+n}}$.

10. Multiply $(a+b)(\sqrt{a}+\sqrt{b})$ by $\sqrt{a}-\sqrt{b}$.
Ans. a^2-b^2.

11. Divide $18\sqrt{18}$ by $2\sqrt{2}$. *Ans.* 27.

12. Divide $\sqrt{54}$ by $\sqrt{2}$. *Ans.* $3\sqrt{3}$.

13. Divide $12\sqrt{3}$ by $3\sqrt[3]{6}$. *Ans.* $4\sqrt[6]{\frac{3}{4}}$.

14. Divide $a\sqrt{b}$ by $c\sqrt[3]{d}$. *Ans.* $\frac{a}{c}\sqrt[6]{\frac{b^3}{d^2}}$.

15. Divide $a\sqrt[m]{b}$ by $c\sqrt[n]{d}$. *Ans.* $\frac{a}{c}\sqrt[mn]{\frac{b^n}{d^m}}$.

16. Divide $\sqrt{\frac{1}{8}}$ by $\sqrt{2}+\sqrt{\frac{9}{2}}$. *Ans.* $\frac{1}{10}$.

17. Divide $\sqrt{\sqrt{\frac{1}{2}}\times\sqrt[3]{24}}$ by $\sqrt{\sqrt[3]{128}\times\sqrt{3}}$.
Ans. $\frac{1}{2}\sqrt[12]{\frac{2}{3}}$.

18. Divide $\sqrt{12}\times\sqrt[3]{4}$ by $\sqrt[4]{\frac{1}{8}}\times\sqrt[3]{3}$. *Ans.* $4\sqrt[12]{288}$.

19. Divide $\sqrt{\frac{a}{b}}$ by $\sqrt[4]{\frac{a}{b}}$. *Ans.* $\sqrt[4]{\frac{a}{b}}$.

20. Divide $(a^2-b^2)\sqrt{a+b}$ by $(a+b)\sqrt[4]{a+b}$.
Ans. $(a-b)\sqrt[4]{a+b}$.

INVOLUTION OF RADICALS.

239. Illustrations.

1. $(a\sqrt[n]{b})^m = a\sqrt[n]{b} \times a\sqrt[n]{b} \times a\sqrt[n]{b} \times \ldots = a^m\sqrt[n]{b^m}.$

2. $(a\sqrt[mn]{b})^m = \left(a\sqrt[m]{\sqrt[n]{b}}\right)^m = a^m\sqrt[n]{b}.$

3. $(a\sqrt[mn]{b})^{mp} = \left[\left(a\sqrt[m]{\sqrt[n]{b}}\right)^m\right]^p = (a^m\sqrt[n]{b})^p = a^{mp}\sqrt[n]{b^p}.$

240. Rule.

1. *If the index of the radical and the exponent of the power are relatively prime, raise the co-efficient to the required power, also the quantity under the radical sign.*

2. *If the index of the radical is a multiple of the exponent of the power, raise the co-efficient to the required power, and divide the index of the radical by the exponent of the power.*

3. *If the index of the radical and the exponent of the power have a common factor, raise the co-efficient to the required power, reject the common factor from the index and exponent, take the remaining factor of the index for the index, and raise the quantity under the radical sign to the power denoted by the remaining factor of the exponent.*

241. Examples.

1. Square $3\sqrt[3]{5}$. *Ans.* $9\sqrt[3]{25}$.

2. Cube $5\sqrt[4]{3}$. *Ans.* $125\sqrt[4]{27}$.

3. Raise $x\sqrt[m]{y}$ to the n^{th} power. *Ans.* $x^n\sqrt[m]{y^n}$.

4. Square $5\sqrt[4]{3}$. *Ans.* $25\sqrt{3}$.

5. Cube $3\sqrt[9]{5}$. *Ans.* $27\sqrt[3]{5}$.

6. Raise $x\sqrt[mn]{y}$ to the n^{th} power. *Ans.* $x^n\sqrt[m]{y}$.

7. Square $7\sqrt[2n]{5}$. *Ans.* $49\sqrt[n]{5}$.

8. Raise $3\sqrt[9]{2}$ to the 6^{th} power. *Ans.* $729\sqrt[3]{4}$.

9. Raise $a\sqrt[18]{b}$ to the 12^{th} power. *Ans.* $a^{12}\sqrt[3]{b^2}$.

10. Raise $x\sqrt[12]{y}$ to the 18^{th} power. *Ans.* $x^{18}\sqrt{y^3}$.

EVOLUTION OF RADICALS.

242. Illustrations.

1. $(a\sqrt[n]{b^m})^r = a^r\sqrt[n]{b^{mr}} \quad \therefore \quad \sqrt[r]{a^r\sqrt[n]{b^{mr}}} = a\sqrt[n]{b^m}$.

2. $(a\sqrt[nr]{b^m})^r = a^r\sqrt[n]{b^m} \quad \therefore \quad \sqrt[r]{a^r\sqrt[n]{b^m}} = a\sqrt[nr]{b^m}$.

3. $(a\sqrt[nr]{b^q})^{pr} = a^{pr}\sqrt[n]{b^{pq}} \quad \therefore \quad \sqrt[pr]{a^{pr}\sqrt[n]{b^{pq}}} = a\sqrt[nr]{b^q}$.

243. Rule.

1. *If the quantity under the radical sign is a perfect power of the degree of the root required, extract the root of the co-efficient, also of the quantity under the radical sign.*

2. *If the index of the required root and the exponent of the quantity under the radical sign are relatively prime, extract the root of the co-efficient, and multiply the index of the given radical by the index of the required root.*

3. *If the index of the required root and the exponent of the quantity under the radical sign have a common factor, extract the required root of the co-efficient, reject the common factor from the index and exponent, multiply the index of the given radical by the remaining factor of the index of the required root, and give to the quantity under the radical sign an exponent equal to the remaining factor of its exponent.*

244. Examples.

1. Extract the square root of $9\sqrt[3]{36}$. *Ans.* $3\sqrt[3]{6}$.

2. Extract the cube root of $125\sqrt{8}$. *Ans.* $5\sqrt{2}$.

3. Extract the n^{th} root of $a^n\sqrt[m]{a^n}$. *Ans.* $a\sqrt[m]{a}$.

4. Extract the square root of $25\sqrt[3]{6}$. *Ans.* $5\sqrt[6]{6}$.

5. Extract the 4^{th} root of $7\sqrt[3]{7}$. *Ans.* $\sqrt[3]{7}$.

6. Extract the n^{th} root of $a\sqrt[m]{b}$. *Ans.* $\sqrt[n]{a}\sqrt[mn]{b}$.

7. Extract the 6^{th} root of $a^{12}\sqrt[19]{c^4}$. *Ans.* $a^2\sqrt[57]{c^2}$.

8. Extract the 4^{th} root of $a^{16}\sqrt[3]{b^2}$. *Ans.* $a^4\sqrt[6]{b}$.

9. Extract the mn^{th} root of $\sqrt[r]{a^{np}}$. *Ans.* $\sqrt[mr]{a^p}$.

10. Extract the 12^{th} root of $x\sqrt{y^9}$. *Ans.* $\sqrt[24]{x^2y^9}$.

11. Extract the 15^{th} root of $\sqrt[3]{(a+b)^{25}}$.

Ans. $\sqrt[9]{(a+b)^5}$.

12. Extract the mn^{th} root of $a^{mn}\sqrt[p]{b^{2mn}} \times \sqrt[r]{c^p} \times \sqrt[q]{d^{md}}$.

Ans. $a\sqrt[p]{b^2} \times \sqrt[mnr]{c^p} \times \sqrt[nq]{d^d}$.

245. Theorems.

1. *No irrational quantity can be expressed by a fraction.*

For, if possible, suppose we have

$$(1) \quad \sqrt[n]{a} = \frac{b}{c}.$$

$$(1)^n = (2) \qquad a = \frac{b^n}{c^n}.$$

Now, if $\frac{b}{c}$ is in its lowest terms, $\frac{b^n}{c^n}$ is in its lowest terms, and we shall have an integral number equal to an irreducible fraction, which is impossible.

Cor. 1. The root of an imperfect power, if carried out decimally, will be interminate; for, if terminate, it could be expressed by a fraction.

Cor. 2. The interminate root can not be a repetend; for, if a repetend, it could be expressed by a fraction.

2. *A quadratic surd can not be equal to the sum of a rational quantity and a quadratic surd.*

For, if possible, suppose we have

$$(1) \quad \sqrt{a} = b + \sqrt{c}.$$

$$(1)^2 = (2) \qquad a = b^2 + 2b\sqrt{c} + c.$$

$$\therefore \quad \sqrt{c} = \frac{a - b^2 - c}{2b}.$$

That is, an irrational quantity is equal to a rational quantity, which is impossible.

3. *If two quadratic surds can not be made similar, their product is irrational.*

Let $\sqrt{a}$ and $\sqrt{b}$ be two such surds. Suppose their product to be rational, and let this product divided by a be equal to r, an entire or a fractional quantity. Then,

$$(1)\quad \sqrt{ab}=ar.$$

$$(1)^2=(2)\qquad ab=a^2r^2.$$

$$(2)\div a=(3)\qquad b=ar^2.$$

$$\sqrt{(3)}=(4)\qquad \sqrt{b}=r\sqrt{a}.$$

That is, $\sqrt{a}$ and $\sqrt{b}$ can be made similar, which is contrary to the supposition. Hence, their product can not be rational, and is therefore irrational.

4. *A quadratic surd can not be equal to the sum of two dissimilar quadratic surds.*

For, if possible, suppose we have

$$(1)\quad \sqrt{a}=\sqrt{b}+\sqrt{c}.$$

$$(1)^2=(2)\qquad a=b+c+2\sqrt{bc}.$$

$$\therefore\quad \sqrt{bc}=\frac{a-b-c}{2}.$$

Now, if $\sqrt{b}$ and $\sqrt{c}$ are in their simplest form, $\sqrt{bc}$ will be irrational; that is, a rational quantity is equal to an irrational quantity, which is impossible.

5. *In an equation of which each member is the sum of a rational quantity and a quadratic surd, the rational quantities of the two members are equal, and also the quadratic surds.*

Let $\qquad (1)\quad a+\sqrt{b}=c+\sqrt{d}.$

If a and c are not equal, let $c = a + n$. Then,

$$(2) \quad a + \sqrt{b} = a + n + \sqrt{d}.$$

$$\therefore \quad (3) \quad \sqrt{b} = n + \sqrt{d},$$

which is impossible by Theorem 2.

$$\therefore \quad a = c; \therefore \quad \sqrt{b} = \sqrt{d}.$$

6. *If* $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

$$(1) \quad \sqrt{a + \sqrt{b}} = x + \sqrt{y}.$$

$$(1)^2 = (2) \quad a + \sqrt{b} = x^2 + y + 2x\sqrt{y}.$$

$$\therefore \quad (3) \quad a = x^2 + y, \text{ and } \sqrt{b} = 2x\sqrt{y}. \quad \text{Th. 5.}$$

$$\therefore \quad (4) \quad a - \sqrt{b} = x^2 - 2x\sqrt{y} + y.$$

$$\sqrt{(4)} = (5) \quad \sqrt{a - \sqrt{b}} = x - \sqrt{y}.$$

7. *If* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

$$(1) \quad \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}.$$

$$(1)^2 = (2) \quad a + \sqrt{b} = x + y + 2\sqrt{xy}.$$

$$\therefore \quad (3) \quad a = x + y \text{ and } \sqrt{b} = 2\sqrt{xy}.$$

$$\therefore \quad (4) \quad a - \sqrt{b} = x - 2\sqrt{xy} + y.$$

$$\sqrt{(4)} = (5) \quad \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

8. *If* $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$, *then* $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$.

$$(1) \quad \sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}.$$

$$(1)^3 = (2) \quad a + \sqrt{b} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}.$$

$$\therefore \quad (3) \quad a = x^3 + 3xy, \text{ and } \sqrt{b} = (3x^2 + y)\sqrt{y}.$$

$$\therefore \quad (4) \quad a - \sqrt{b} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}.$$

$$\sqrt[3]{(4)} = (5) \quad \sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}.$$

246. Problem.

To extract the square root of $a \pm \sqrt{b}$.

Let (1) $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$.

Then (2) $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$. Th. 7.

$(1)^2 =$ (3) $x+2\sqrt{xy}+y=a+\sqrt{b}$.

$\therefore$ (4) $x+y=a$. Th. 5.

$(1)\times(2) =$ (5) $x-y=\sqrt{a^2-b}$.

$\sqrt{\dfrac{(4)+(5)}{2}} =$ (6) $\sqrt{x}=\pm\sqrt{\dfrac{a+\sqrt{a^2-b}}{2}}$.

$\sqrt{\dfrac{(4)-(5)}{2}} =$ (7) $\sqrt{y}=\pm\sqrt{\dfrac{a-\sqrt{a^2-b}}{2}}$.

Adding the (6) and (7), and substituting $\sqrt{a+\sqrt{b}}$ for $\sqrt{x}+\sqrt{y}$, we have

$$\sqrt{a+\sqrt{b}}=\pm\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

Subtracting the (7) from the (6), and substituting $\sqrt{a-\sqrt{b}}$ for $\sqrt{x}-\sqrt{y}$, we have

$$\sqrt{a-\sqrt{b}}=\pm\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\mp\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

247. Examples.

1. Extract the square root of $16+6\sqrt{7}$.
C. A. 15.

SOLUTION.

$$\sqrt{16+6\sqrt{7}}=\sqrt{16+\sqrt{252}},$$

$$\therefore \quad a=16,\ b=252,\ \sqrt{a^2-b}=\sqrt{256-252}=2.$$

$$\therefore \quad \sqrt{16+6\sqrt{7}}=\pm\sqrt{\frac{16+2}{2}}\pm\sqrt{\frac{16-2}{2}}=\pm 3\pm\sqrt{7}.$$

2. Extract the square root of $21+8\sqrt{5}$.

Ans. $\pm 4\pm\sqrt{5}$.

3. Extract the square root of $67-16\sqrt{3}$.

Ans. $\pm 8\mp\sqrt{3}$.

4. Extract the square root of $8+2\sqrt{15}$.

Ans. $\pm\sqrt{5}\pm\sqrt{3}$.

5. Extract the square root of $12-2\sqrt{35}$.

Ans. $\pm\sqrt{7}\mp\sqrt{5}$.

6. Extract the square root of $131-22\sqrt{10}$.

Ans. $\pm 11\mp\sqrt{10}$.

7. Extract the square root of $30+12\sqrt{6}$.

Ans. $\pm 3\sqrt{2}\pm 2\sqrt{3}$.

8. Extract the square root of $p^2q+pq^2+2pq\sqrt{pq}$.

Ans. $\pm p\sqrt{q}\pm q\sqrt{p}$.

FRACTIONAL AND NEGATIVE EXPONENTS.

248. Illustrations.

$$(1) \quad a^{\frac{1}{n}}=\sqrt[n]{a}.$$

$$(1)^m=(2) \quad a^{\frac{m}{n}}=\sqrt[n]{a^m}.$$

The numerator of the fractional exponent denotes the power to which the quantity is to be raised; and the denominator, the root to be extracted.

It now remains to be proved that we can perform the processes of multiplication, division, involution, and evolution of radical quantities, if we substitute for the radical sign and index the equivalent fractional exponent.

1*st.* In multiplication, we have

1. $\sqrt[n]{a^m} \times \sqrt[q]{a^p} = \sqrt[nq]{a^{mq+np}}.$

But $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, and $\sqrt[q]{a^p} = a^{\frac{p}{q}}$,

and $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{mq+np}{nq}} = \sqrt[nq]{a^{mq+np}}.$

2. $\sqrt[n]{a^m} \times \sqrt[q]{\frac{1}{a^p}} = \sqrt[nq]{\frac{a^{mq}}{a^{np}}} = \sqrt[nq]{a^{mq-np}}.$

But $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, and $\sqrt[q]{\frac{1}{a^p}} = \frac{1}{a^{\frac{p}{q}}} = a^{-\frac{p}{q}}$,

and $a^{\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{\frac{mq-np}{nq}} = \sqrt[nq]{a^{mq-np}}.$

3. $\sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{\frac{1}{a^p}} = \sqrt[nq]{\frac{1}{a^{mq+np}}} = \sqrt[nq]{a^{-(mq+np)}}.$

But $\sqrt[n]{\frac{1}{a^m}} = \frac{1}{a^{\frac{m}{n}}} = a^{-\frac{m}{n}}$, and $\sqrt[q]{\frac{1}{a^p}} = \frac{1}{a^{\frac{p}{q}}} = a^{-\frac{p}{q}}$,

and $a^{-\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{-\frac{mq+np}{nq}} = \sqrt[nq]{a^{-(mq+np)}}.$

2*d.* Similar results can be obtained for division.

3*d*. In involution, we have

$$(\sqrt[n]{a^m})^p = \sqrt[n]{a^{mp}}; \ \sqrt[n]{a^m} = a^{\frac{m}{n}},$$

and
$$(a^{\frac{m}{n}})^p = a^{\frac{mp}{n}} = \sqrt[n]{a^{mp}}.$$

4*th*. In evolution, we have

$$\sqrt[r]{\sqrt[n]{a^m}} = \sqrt[nr]{a^m}; \text{ but } \sqrt[n]{a^m} = a^{\frac{m}{n}},$$

and
$$\sqrt[r]{a^{\frac{m}{n}}} = a^{\frac{m}{nr}} = \sqrt[nr]{a^m}.$$

TO RENDER RADICALS RATIONAL.

249. Case I.

To render Monomial Radicals Rational.

1. $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a.$

2. $\sqrt[3]{a} \times \sqrt[3]{a^2} = \sqrt[3]{a^3} = a.$

3. $\sqrt{a^n} \times \sqrt[m]{a^{m-n}} = \sqrt[m]{a^m} = a.$

250. Rule.

Multiply the given radical by another radical of the same degree, having the same quantity under the radical sign, with an exponent equal to the index of the given radical minus the exponent of the quantity under the radical sign.

251. Examples.

1. What multiplier will make $\sqrt[5]{a^3}$ rational?

Ans. $\sqrt[5]{a^2}$.

2. What multiplier will make $\sqrt[7]{a^4}$ rational?

Ans. $\sqrt[7]{a^3}$.

3. What multiplier will make $\sqrt[r]{a^p}$ rational?

Ans. $\sqrt[r]{a^{r-p}}$.

4. Reduce $\frac{3}{\sqrt{5}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{3\sqrt{5}}{5}$.

5. Reduce $\frac{a}{\sqrt[3]{b}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{a\sqrt[3]{b^2}}{b}$

6. Reduce $\frac{2}{\sqrt[3]{3}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{2\sqrt[3]{9}}{3}$.

252. Case II.

To render Binomial Radicals Rational.

1. What multiplier will make $\sqrt[m]{a^p} - \sqrt[n]{b^q}$ rational?

Let l be the l. c. m. of m and n, then $(\sqrt[m]{a^p})^l - (\sqrt[n]{b^q})^l$ is rational; but, by Art. 56, $(\sqrt[m]{a^p})^l - (\sqrt[n]{b^q})^l$ is divisible by $\sqrt[m]{a^p} - \sqrt[n]{b^q}$. The quotient will be the multiplier sought, since the product of the divisor and quotient equals the dividend.

2. What multiplier will make $\sqrt[m]{a^p} + \sqrt[n]{b^q}$ rational?

Let l be the l. c. m. of m and n. Then $(\sqrt[m]{a^p})^l - (\sqrt[n]{b^q})^l$ is rational, and divisible by $\sqrt[m]{a^p} + \sqrt[n]{b^q}$, if l is even, Art. 59; and $(\sqrt[m]{a^p})^l + (\sqrt[n]{b^q})^l$ is rational, and divisible by $\sqrt[m]{a^p} + \sqrt[n]{b^q}$, if l is odd, Art. 61. In either case, the quotient will be a multiplier which will make $\sqrt[m]{a^p} + \sqrt[n]{b^q}$ rational.

It will often be convenient to use the fractional exponent for the radical sign.

253. Examples.

1. What multiplier will make $\sqrt{7} - \sqrt{5}$ rational?

Ans. $\sqrt{7} + \sqrt{5}$.

2. What multiplier will make $\sqrt{11} + \sqrt{7}$ rational?

Ans. $\sqrt{11} - \sqrt{7}$.

3. What multiplier will make $\sqrt[3]{a} - \sqrt[3]{b}$ rational?

Ans. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

4. What multiplier will make $\sqrt[7]{a^3} - \sqrt[5]{b^2}$ rational?

Ans. $a^{\frac{102}{7}} + a^{\frac{99}{7}}b^{\frac{2}{5}} + a^{\frac{96}{7}}b^{\frac{4}{5}} + \ldots + a^{\frac{3}{7}}b^{\frac{66}{5}} + b^{\frac{68}{5}}$.

5. What multiplier will make $\sqrt[6]{a^5} + \sqrt[4]{b^3}$ rational?

Ans. $a^{\frac{55}{6}} - a^{\frac{50}{6}}b^{\frac{3}{4}} + a^{\frac{45}{6}}b^{\frac{6}{4}} - \ldots + a^{\frac{5}{6}}b^{\frac{30}{4}} - b^{\frac{33}{4}}$.

254. Case III.

To render Trinomial Radicals Rational.

A trinomial quadratic surd can be rendered rational by two multiplications. Thus,

$$(\sqrt{p}+\sqrt{q}-\sqrt{r})(\sqrt{p}+\sqrt{q}+\sqrt{r})=p+2\sqrt{pq}+q-r$$
$$=p+q-r+2\sqrt{pq}=2\left(\frac{p+q-r}{2}+\sqrt{pq}\right),$$

and

$$2\left(\frac{p+q-r}{2}+\sqrt{pq}\right)\left(\frac{p+q-r}{2}-\sqrt{pq}\right)=\frac{(p+q-r)^2}{2}-2pq.$$

255. Examples.

Render the denominators of the following fractions rational.

1. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. *Ans.* $\dfrac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12}$.

2. $\dfrac{1}{\sqrt{6}+\sqrt{7}-\sqrt{5}}$.

Ans. $\dfrac{\sqrt{252}+\sqrt{294}+\sqrt{210}-4(\sqrt{6}+\sqrt{7}+\sqrt{5})}{52}$.

IMAGINARY QUANTITIES.

ADDITION AND SUBTRACTION.

256. Examples.

1. $\sqrt{-p}\pm\sqrt{-q}=\sqrt{p}\,.\sqrt{-1}\pm\sqrt{q}\,.\sqrt{-1}$
$$=(\sqrt{p}\pm\sqrt{q})\sqrt{-1}.$$

2. Find the sum of $\sqrt{-9}$ and $\sqrt{-16}$.

Ans. $7\sqrt{-1}$.

3. Find the difference of $\sqrt{-50}$ and $\sqrt{-18}$.

Ans. $2\sqrt{-2}$.

4. Find the sum of $\sqrt[4]{-81}$ and $\sqrt[4]{-16}$.

Ans. $5\sqrt[4]{-1}$.

5. Find the difference of $\sqrt[4]{-p}$ and $\sqrt[4]{-q}$.

Ans. $(\sqrt[4]{p}-\sqrt[4]{q})\sqrt[4]{-1}$.

MULTIPLICATION AND DIVISION.

257. Illustrations.

It would seem from the rule for the multiplication of radicals that $\sqrt{-a}\times\sqrt{-a}=\sqrt{a^2}=\pm a$; but the plus value is to be rejected, since, by definition, $(\sqrt{-a})^2=-a$. This is also shown thus,

$$\sqrt{-a}\times\sqrt{-a}=\sqrt{a}.\sqrt{-1}\times\sqrt{a}.\sqrt{-1}=a\times-1=-a.$$

Also,

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{a}.\sqrt{-1}\times\sqrt{b}.\sqrt{-1}$$

$$=\sqrt{ab}\times-1=-\sqrt{ab}.$$

Also,

$$\sqrt{-a}\div\sqrt{-b}=\sqrt{a}.\sqrt{-1}\div\sqrt{b}.\sqrt{-1}=\sqrt{\frac{a}{b}}.$$

258. Examples.

1. Multiply $5\sqrt{-5}$ by $3\sqrt{-3}$. *Ans.* $-15\sqrt{15}$.

2. Multiply $6\sqrt{-5}$ by $8\sqrt{-20}$. *Ans.* -480.

3. Divide $8\sqrt{-3}$ by $3\sqrt{-4}$. *Ans.* $\frac{4}{3}\sqrt{3}$.

4. Divide $-9\sqrt{-1}$ by $-2\sqrt{-4}$. *Ans.* $\frac{9}{4}$.

5. Multiply $3+2\sqrt{-2}$ by $2-2\sqrt{-3}$.

Ans. $6+4\sqrt{-2}-6\sqrt{-3}+4\sqrt{6}$.

INVOLUTION.

259. Illustrations.

1. $(\sqrt{-1})^0 = 1$.
2. $(\sqrt{-1})^1 = \sqrt{-1}$.
3. $(\sqrt{-1})^2 = -1$.
4. $(\sqrt{-1})^3 = -\sqrt{-1}$.
5. $(\sqrt{-1})^{4n} = \{[(\sqrt{-1})^2]^2\}^n = [(-1)^2]^n = 1^n = 1$.
6. $(\sqrt{-1})^{4n+1} = (\sqrt{-1})^{4n}.(\sqrt{-1})^1 = 1.\sqrt{-1} = \sqrt{-1}$.
7. $(\sqrt{-1})^{4n+2} = (\sqrt{-1})^{4n}.(\sqrt{-1})^2 = 1.-1 = -1$.
8. $(\sqrt{-1})^{4n+3} = (\sqrt{-1})^{4n}.(\sqrt{-1})^3$
   $= 1.-\sqrt{-1} = -\sqrt{-1}$.

By comparing the 5*th*, 6*th*, 7*th*, 8*th* with the 1*st*, 2*d*, 3*d*, 4*th*, respectively, we find that the powers are the same.

Since $4n$, $4n+1$, $4n+2$, $4n+3$ may be made to represent all positive whole numbers, by giving to n in succession the values 0, 1, 2, 3, 4, etc., and since the remainders obtained by dividing these respectively by 4 are 0, 1, 2, 3, we have for the involution of $\sqrt{-1}$ the rule:

260. Rule.

Divide the exponent by 4, and take that power of $\sqrt{-1}$ *indicated by the remainder.*

261. Examples.

1. Raise $\sqrt{-1}$ to the 20^{th} power. *Ans.* 1.

2. Raise $\sqrt{-1}$ to the 29^{th} power. *Ans.* $\sqrt{-1}$.

3. Raise $\sqrt{-1}$ to the 34^{th} power. *Ans.* -1.

4. Raise $\sqrt{-1}$ to the 99^{th} power. *Ans.* $-\sqrt{-1}$.

5. Raise $-a\sqrt{-1}$ to the 7^{th} power. *Ans.* $a^7\sqrt{-1}$.

6. Raise $p+\sqrt{-1}$ to the 4^{th} power.

Ans. $p^4-6p^2+1+4p(p^2-1)\sqrt{-1}$.

EVOLUTION.

262. Theorems.

1. *If* $\sqrt{a+\sqrt{-b}}=\sqrt{x}+\sqrt{-y}$, *then* $\sqrt{a-\sqrt{-b}}$ $=\sqrt{x}-\sqrt{-y}$.

$$(1) \quad \sqrt{a+\sqrt{-b}}=\sqrt{x}+\sqrt{-y}.$$

$$(1)^2=(2) \quad a+\sqrt{-b}=x+2\sqrt{-xy}-y.$$

$$\therefore \quad (3) \quad a=x-y \text{ and } \sqrt{-b}=2\sqrt{-xy}.$$

$$\therefore \quad (4) \quad a-\sqrt{-b}=x-2\sqrt{-xy}-y.$$

$$\sqrt{(4)}=(5) \quad \sqrt{a-\sqrt{-b}}=\sqrt{x}-\sqrt{-y}.$$

2. *If* $\sqrt[3]{a+\sqrt{-b}} = x + \sqrt{-y}$, *then* $\sqrt[3]{a-\sqrt{-b}} = x - \sqrt{-y}$.

$$(1) \quad \sqrt[3]{a+\sqrt{-b}} = x + \sqrt{-y}.$$

$$(1)^3 = (2) \quad a + \sqrt{-b} = x^3 + 3x^2\sqrt{-y} - 3xy - y\sqrt{y}.$$

$$\therefore \quad (3) \quad a = x^3 - 3xy \text{ and } \sqrt{-b} = (3x^2 - y)\sqrt{y}.$$

$$\therefore \quad (4) \quad a - \sqrt{-b} = x^3 - 3x^2\sqrt{-y} - 3xy + y\sqrt{-y}.$$

$$\sqrt[3]{(4)} = (5) \quad \sqrt[3]{a-\sqrt{-b}} = x - \sqrt{-y}.$$

263. Problem.

To extract the square root of $a \pm \sqrt{-b}$.

Let us assume

$$(1) \quad \sqrt{x} + \sqrt{-y} = \sqrt{a+\sqrt{-b}}.$$

Then,

$$(2) \quad \sqrt{x} - \sqrt{-y} = \sqrt{a-\sqrt{-b}}.$$

$$(1)^2 = (3) \quad x - y + 2\sqrt{-xy} = a + \sqrt{-b}.$$

$$\therefore \quad (4) \quad x - y = a.$$

$$(1) \times (2) = (5) \quad x + y = \sqrt{a^2 + b}.$$

$$\sqrt{\frac{(4)+(5)}{2}} = (6) \quad \sqrt{x} = \pm\sqrt{\frac{a+\sqrt{a^2+b}}{2}}.$$

$$\sqrt{\frac{(4)-(5)}{2}} = (7) \quad \sqrt{-y} = \pm\sqrt{\frac{a-\sqrt{a^2+b}}{2}}.$$

Hence,

$$\sqrt{a+\sqrt{-b}}=\pm\sqrt{\frac{a+\sqrt{a^2+b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2+b}}{2}},$$

and

$$\sqrt{a-\sqrt{-b}}=\pm\sqrt{\frac{a+\sqrt{a^2+b}}{2}}\mp\sqrt{\frac{a-\sqrt{a^2+b}}{2}}.$$

264. Examples.

1. Extract the square root of $6+6\sqrt{-3}$.

   Ans. $\pm 3\pm\sqrt{-3}$.

2. Extract the square root of $-2\sqrt{-1}$.

   Ans. $\pm 1\mp\sqrt{-1}$.

3. Extract the square root of $23-10\sqrt{-2}$.

   Ans. $\pm 5\mp\sqrt{-2}$.

4. Extract the square root of $1+56\sqrt{-3}$.

   Ans. $\pm 7\pm 4\sqrt{-3}$.

5. Extract the square root of $a^2-2ab+2(a-b)\sqrt{-b^2}$.

   Ans. $a-b+b\sqrt{-1}$.

265. Miscellaneous Examples in Radicals.

1. Simplify $\sqrt{3a^2-6ab+3b^2}$. *Ans.* $(a-b)\sqrt{3}$.

2. Simplify $\sqrt{45a^7-60a^5b^3+20a^3b^6}$.

   Ans. $(3a^2-2b^3)\,a\sqrt{5a}$.

3. Add $\sqrt{4a^3b}$ and $\sqrt{a^3b^3}$.

Ans. $a(2+b)\sqrt{ab}$.

4. Subtract $\sqrt{\frac{48}{25}}$ from $\sqrt{\frac{27}{16}}$. *Ans.* $-\frac{1}{20}\sqrt{3}$.

5. Divide $\sqrt{2} \times \sqrt[3]{3}$ by $\sqrt[4]{4} \times \sqrt[6]{6}$. *Ans.* $\sqrt[6]{\frac{3}{2}}$.

6. Raise $p\sqrt[12]{q}$ to the 9^{th} power. *Ans.* $p^9\sqrt[4]{q^3}$.

7. Extract the 4^{th} root of $p^8\sqrt[6]{q^2}$. *Ans.* $p^2\sqrt[12]{q}$.

8. Reduce $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$.

Ans. $\pm 2\sqrt{bc-b^2}$.

9. Reduce $\sqrt{4a^2+4\sqrt{a^4-b^4}}-\sqrt{4a^2-4\sqrt{a^4-b^4}}$.

Ans. $\pm 2\sqrt{2a^2-2b^2}$.

10. Reduce $\dfrac{2\sqrt{2}\times\sqrt[3]{3}}{\frac{1}{2}\sqrt{2}}$. *Ans.* $4\sqrt[3]{3}$.

11. Reduce $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}\,(3)^{\frac{1}{2}}}\right\}^4$. *Ans.* $\dfrac{1}{384}\sqrt[3]{3}$.

12. Reduce $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{3\frac{1}{2}}}{2\sqrt{2}\,(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$.

Ans. $\sqrt[4]{\frac{1}{6}\left(\frac{1}{8}\sqrt{6}+\sqrt{21}\right)}$.

13. Reduce $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$.

Ans. ± 10.

INEQUATIONS.

266. Definitions.

1. **An inequation** is the expression of the inequality of two quantities. Thus, $7 > 5$ and $3 < 4$.

2. Two inequations subsist in the same sense when the greater is at the left in both, or at the right in both. Thus, $13 > 9$ and $10 > 8$; also, $4 < 6$ and $3 < 5$.

267. Propositions.

1. *If the same quantity be added to both members of an inequation, the resulting inequation will subsist in the same sense.*

Thus, $5 > 3$ and $5 + 2 > 3 + 2$, or $7 > 5$.

2. *If the same quantity be subtracted from both members of an inequation, the resulting inequation will subsist in the same sense.*

Thus, $7 < 9$ and $7 - 2 < 9 - 2$, or $5 < 7$.

Also, $8 > 5$ and $8 - 10 > 5 - 10$ or $-2 > -5$.

These propositions enable us to transpose. Thus,

$$2x - 8 > x + 3, \quad \therefore \quad 2x - x > 8 + 3, \text{ or } x > 11.$$

3. *If both members of an inequation be multiplied by the same positive quantity, the resulting inequation will subsist in the same sense.*

Thus, $8 > 5$ and $8 \times 3 > 5 \times 3$, or $24 > 15$;

and

$$-5 < -3 \text{ and } -5 \times 2 < -3 \times 2, \text{ or } -10 < -5.$$

This proposition enables us to clear an inequation of fractions. Thus,

$$\tfrac{3}{4}x > \tfrac{1}{2}a, \quad \therefore \quad 3x > 2a.$$

4. *If both members of an inequation be divided by the same positive quantity, the resulting inequation will subsist in the same sense.*

Thus, $8 > 4$ and $8 \div 2 > 4 \div 2$, or $4 > 2$;

and

$$-15 < -12 \text{ and } -15 \div 3 < -12 \div 3, \text{ or } -5 < -4.$$

This proposition enables us to reduce an inequation by division. Thus,

$$3x > 6a, \quad \therefore \quad x > 2a.$$

5. *If both members of an inequation be multiplied or divided by the same negative quantity, the resulting inequation will subsist in a contrary sense.* Thus,

$$8 > 6 \text{ and } 8 \div -2 < 6 \div -2, \text{ or } -4 < -3;$$

and

$$-5 < 3 \text{ and } -5 \times -3 > 3 \times -3, \text{ or } 15 > -9.$$

6. *If the signs of both members of an inequation be changed, the sense will be changed;* for changing the signs is equivalent to multiplying or dividing by -1.

7. *If two inequations which subsist in the same sense be added, member to member, the resulting inequation will subsist in the same sense.* Thus,

$$5 < 8 \text{ and } -7 < -3, \text{ and } 5 - 7 < 8 - 3, \text{ or } -2 < 5.$$

8. *If one inequation be subtracted from another which subsists in the same sense, the resulting inequation may or may not subsist in the same sense.* Thus,

$$8 > 5 \text{ and } 3 > 2, \text{ and } 8 - 3 > 5 - 2, \text{ or } 5 > 3.$$

Also,

$$8 > 5 \text{ and } 7 > 2, \text{ and } 8 - 7 < 5 - 2, \text{ or } 1 < 3.$$

This operation, then, is to be avoided unless the sense can be determined.

9. *If both members of an inequation of positive quantities be raised to the same power, the resulting inequation will subsist in the same sense.* Thus,

$$5 > 3 \text{ and } 5^3 > 3^3, \text{ or } 125 > 27.$$

10. *If both members of an inequation of negative quantities be raised to an even power, the resulting inequation will subsist in a contrary sense; but if both members be raised to an odd power, the resulting inequation will subsist in the same sense.* Thus,

$$-3 > -5,\ (-3)^2 < (-5)^2 \text{ or } 9 < 25,$$
$$\text{and } (-3)^3 > (-5)^3 \text{ or } -27 > -125.$$

11. *If the same root be extracted of both members of an inequation of positive quantities, or the same odd root of an inequa-*

tion of negative quantities, the resulting inequation will subsist in the same sense. Thus,

$$27 > 8 \text{ and } \sqrt[3]{27} > \sqrt[3]{8}, \text{ or } 3 > 2; \ -64 < -27,$$

$$\text{and } \sqrt[3]{-64} < \sqrt[3]{-27}, \text{ or } -4 < -3.$$

268. Examples.

1. Given $2x + \frac{1}{2}x - 4 > 6$, to find the limit of x.

Ans. $x > 4$.

2. Given $\left\{\begin{array}{l} \frac{x}{b} + b < \frac{x}{a} + a, \\ \frac{x}{b} - \frac{x}{a} > \frac{a-b}{b}, \end{array}\right\}$ to find the limits of x.

Ans. $x < ab$ and $x > a$.

3. Prove that $a^2 + b^2 > 2ab$, if $a > b$ or $a < b$.

4. Prove that $a^2 + b^2 = 2ab$, if $a = b$.

5. Prove that any positive fraction whose terms are unequal, plus its reciprocal, is greater than 2.

6. Prove that $x^2 + y^2 + z^2 > xy + xz + yz$, if x, y, z are unequal.

7. Prove that $x^3 + 1 > x^2 + x$, if $x > 1$ or $x < 1$.

8. Prove that $\frac{m+p+r}{n+q+s} > \frac{m}{n}$ and $< \frac{r}{s}$, if $\frac{m}{n} < \frac{p}{q} < \frac{r}{s}$.

9. Prove that $pqr > (p+q-r)(p+r-q)(q+r-p)$, if p, q, r are unequal.

EQUATIONS OF THE SECOND DEGREE.

269. Definitions and Classification.

1. **An equation of the second degree involving but one unknown quantity** is an equation in which the greatest exponent of the unknown quantity is 2.

2. **A complete equation of the second degree** involves both the first and the second power of the unknown quantity. Thus, $ax^2 + bx = c$ is a complete equation.

3. **An incomplete equation of the second degree** involves only the second power of the unknown quantity. Thus, $ax^2 = c$ is an incomplete equation.

4. **An equation of the second degree involving two or more unknown quantities** is an equation in which the greatest sum of the exponents of the unknown quantities in any of the terms is 2. Thus, $xy + bz + yz = q$.

5. Equations of the second degree are also called **quadratic equations**—pure quadratics, when incomplete; affected quadratics, when complete.

INCOMPLETE EQUATIONS.

270. Form.

Every incomplete equation of the second degree can, by clearing of fractions, transposing, reducing, and changing the signs, if necessary, be placed under the form

$$x^2 = q.$$

1ST SOLUTION.

$$(1) \quad x^2 = q.$$

$$\sqrt{(1)} = (2) \quad x = \pm\sqrt{q}.$$

2D SOLUTION.

$$(1) \quad x^2 = q.$$

$$\therefore \quad (2) \quad x^2 - q = 0.$$

$$\text{Or,} \quad (3) \quad (x - \sqrt{q})(x + \sqrt{q}) = 0.$$

Equation (3) will be satisfied

$$\text{If} \begin{cases} \text{either } x - \sqrt{q} = 0, & \therefore \quad x = +\sqrt{q}; \\ \text{or} \quad x + \sqrt{q} = 0, & \therefore \quad x = -\sqrt{q}. \end{cases}$$

VERIFICATION.

$$(+\sqrt{q})^2 = q \text{ and } (-\sqrt{q})^2 = q.$$

271. Rule.

Reduce the equation to the form $x^2 = q$, *extract the square root of both members, giving to the second member the double sign* $\pm$.

272. Examples.

1. Given $5x^2 - \frac{3}{7}x^2 = 224$, to find x. *Ans.* $x = \pm 7$.

2. Given $6x^2 - 2x^2 = \frac{25}{9}$, to find x. *Ans.* $x = \pm\frac{5}{6}$.

3. Given $5x^2 - 7x^2 = -18$, to find x. *Ans.* $x = \pm 3$.

4. Given $ax^2 = b$, to find x. *Ans.* $x = \pm \sqrt{\frac{a}{b}}$.

5. Given $\frac{a}{b}x^2 - \frac{c}{d} = \frac{e}{f}x^2 + g$, to find x.

Ans. $x = \pm \sqrt{\frac{bdfg + bcf}{adf - bde}}$.

6. Given $\frac{m}{x+n} - \frac{m}{x-n} = p$, to find x.

Ans. $x = \pm \sqrt{\frac{n(pn - 2m)}{p}}$.

7. Given $\frac{\sqrt{a^2 - p^2x^2}}{x} = q$, to find x.

Ans. $x = \pm \frac{a}{\sqrt{p^2 + q^2}}$.

8. What two numbers are to each other as 5 to 7, and the difference of whose squares is 216? *Ans.* 15 and 21.

9. What two numbers are to each other as m to n, and the sum of whose squares is s^2?

Ans. $\pm \frac{ms}{\sqrt{m^2 + n^2}}$ and $\pm \frac{ns}{\sqrt{m^2 + n^2}}$.

10. Divide 21 into two such parts that the quotient of the less divided by the greater shall be to the quotient of the greater divided by the less as 9 to 16. *Ans.* 9 and 12.

11. Divide n into two such parts that the quotient of the first divided by the second shall be to the quotient of the second divided by the first as p to q.

Ans. $\frac{n\sqrt{p}}{\sqrt{p} + \sqrt{q}}$ and $\frac{n\sqrt{q}}{\sqrt{p} + \sqrt{q}}$.

12. A rectangular field whose sides are to each other as 3 to 5 contains 6 acres. What are the sides?

Ans. 24 *rd.* and 40 *rd.*

273. Analogous Forms.

1. $$(1)\quad x^n = q.$$

$$\sqrt[n]{(1)} = (2)\quad x = \sqrt[n]{q}.$$

If n is even, the double sign $\pm$ must be used.

2. $$(1)\quad \frac{28x}{x+18} = \frac{63\,(x+18)}{4x}.$$

$$(1) \times 4x\,(x+18) = (2)\quad 112x^2 = 63\,(x+18)^2.$$

$$(2) \div 7 = (3)\quad 16x^2 = 9\,(x+18)^2.$$

$$\sqrt{(3)} = (4)\quad 4x = 3\,(x+18).$$

$$\text{Or,}\quad 4x = 3x + 54.$$

$$\therefore\quad x = 54.$$

3. Given $(x-a)^2 = b$, to find x. *Ans.* $x = a \pm \sqrt{b}$.

4. Given $x^2 - 2ax + a^2 = b^2$, to find x.

Ans. $x = a \pm b$.

5. Given $x^2 + 2ax = b^2 - a^2$, to find x.

Ans. $x = \pm b - a$.

6. Given $px^2 + pq = 2px\sqrt{q} + qx^2$, to find x.

Ans. $x = \dfrac{\sqrt{pq}}{\sqrt{p} \mp \sqrt{q}}$.

7. Given $7x^3 = 875$, to find x. *Ans.* $x = 5$.

8. The volume of a rectangular box is 51840 cu. in.; the length, breadth, and depth are to each other as 5, 3, and 2. Find these dimensions. *Ans.* 60, 36, and 24.

9. A and B, starting from different places, travel toward each other; and, on meeting, it appears that A has traveled 30 miles more than B, and that it would take A 4 days to travel B's distance, and B 9 days to travel A's distance. How far has each traveled?

Ans. B 60 miles, A 90 miles.

COMPLETE EQUATIONS.

274. Forms.

Every complete equation of the second degree can, by clearing of fractions, transposing, reducing, and changing the signs, if necessary, be placed under one of the following forms:

$$(1) \quad x^2 + 2px = q.$$

$$(2) \quad x^2 - 2px = q.$$

$$(3) \quad x^2 + 2px = -q.$$

$$(4) \quad x^2 - 2px = -q.$$

275. Solution of the First Form.

$$x^2 + 2px = q.$$

Adding p^2 to both members, we have

$$x^2 + 2px + p^2 = q + p^2.$$

Extracting the square root, we have

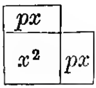
$$x + p = \pm\sqrt{q + p^2}.$$

Transposing p, we have

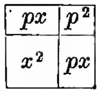
$$x = -p \pm \sqrt{q + p^2}.$$

Adding the square of half the co-efficient of the first power of the unknown quantity to both members is called *completing the square,* since it renders the first member a perfect square. It may be illustrated geometrically thus:

$x^2 + 2px = q$



$x^2 + 2px + p^2 = q + p^2$



276. Solution of the Second Form.

$$x^2 - 2px = q.$$

$$x^2 - 2px + p^2 = q + p^2.$$

$$x - p = \pm\sqrt{q + p^2}.$$

$$x = p \pm \sqrt{q + p^2}.$$

277. Solution of the Third Form.

$$x^2 + 2px = -q.$$

$$x^2 + 2px + p^2 = -q + p^2.$$

$$x + p = \pm\sqrt{-q + p^2}.$$

$$x = -p \pm \sqrt{-q + p^2}.$$

278. Solution of the Fourth Form.

$$x^2 - 2px = -q.$$

$$x^2 - 2px + p^2 = -q + p^2.$$

$$x - p = \pm \sqrt{-q + p^2}.$$

$$x = p \pm \sqrt{-q + p^2}.$$

By comparing the values of x in each of these forms with the equation from which they are deduced, we have the rule:

279. Rule.

Reduce the equation to one of the four forms, then write the unknown quantity equal to half the co-efficient of the first power of the unknown quantity with its sign changed, plus or minus the square root of the sum of the second member and the square of half the co-efficient of the first power of the unknown quantity.

280. Examples.

1. Given $x^2 + 6x = 55$, to find x.

SOLUTION IN FULL.

$$x^2 + 6x = 55.$$

$$x^2 + 6x + 9 = 64.$$

$$x + 3 = \pm 8.$$

$$x = -3 \pm 8 = 5 \text{ or } -11.$$

SOLUTION BY THE RULE.

$$x^2 + 6x = 55.$$

$$x = -3 \pm \sqrt{55 + 9}.$$

$$x = -3 \pm 8 = 5 \text{ or } -11.$$

2. Given $x^2 + 8x = 48$, to find x.

Ans. $x = 4$ or -12.

3. Given $x^2 - 14x = 51$, to find x.

Ans. $x = 17$ or -3.

4. Given $x^2 + 10x = -24$, to find x.

Ans. $x = -4$ or -6.

5. Given $x^2 - 10x = -21$, to find x.

Ans. $x = 7$ or 3.

By comparing the answers with the forms of the equations from which they are deduced, we see that in the first form the values of x have unlike signs, the negative being numerically the greater; that in the second form, the values of x have unlike signs, the positive being the greater; that in the third form, the values of x have like signs, both being negative; and that in the fourth form, the values of x have like signs, both being positive.

We also see that in each form the sum of the values of x is equal to the co-efficient of x with its sign changed, and that the product of these values is equal to the second member with its sign changed.

The theory of these interesting facts will shortly be given.

6. Given $12x^2 - 24x = 420$, to find x.

Ans. $x = 7$ or -5.

7. Given $x^2 - 35x = -300$, to find x.

Ans. $x = 20$ or 15.

8. Given $5x^2 + 8x = 21$, to find x.

Ans. $x = \frac{7}{5}$ or -3.

9. Given $7x^2 - 5x = 522$, to find x.

Ans. $x = 9$ or $-8\frac{2}{7}$.

10. Given $4x^2 + 3x = 115$, to find x.

Ans. $x = 5$ or $-5\frac{3}{4}$.

11. Given $x^2 - \frac{76}{15}x = -\frac{77}{15}$, to find x.

Ans. $x = \frac{11}{3}$ or $\frac{7}{5}$.

12. Given $3x^2 + 10x = -\frac{16}{3}$, to find x.

Ans. $x = -\frac{2}{3}$ or $-\frac{8}{3}$.

13. Given $12x^2 - 187x = -481$, to find x.

Ans. $x = \frac{37}{3}$ or $\frac{13}{4}$.

14. Given $x^2 - \dfrac{a^2 + b^2}{ab}x = -1$, to find x.

Ans. $x = \dfrac{a}{b}$ or $\dfrac{b}{a}$.

15. Given $ax^2 - (a^2 - b)x = ab$, to find x.

Ans. $x = a$ or $-\dfrac{b}{a}$.

16. Given $ax^2 - ax + \dfrac{ab}{a-b} = bx^2 + bx$, to find x.

Ans. $x = \dfrac{a}{a-b}$ or $\dfrac{b}{a-b}$.

17. Given $px^2 - px = \dfrac{pq}{p+q} - qx^2 - qx$, to find x.

Ans. $x = \dfrac{p}{p+q}$ or $-\dfrac{q}{p+q}$.

18. Given $\dfrac{a^2 - b^2}{b^2}x^2 + 2nx = m^2 + n^2$, to find x.

Ans. $x = \dfrac{b}{b^2 - a^2}\left(bn \pm \sqrt{a^2m^2 + a^2n^2 - b^2m^2}\right)$.

281. Hindoo Formula for Solving Quadratics.

$$(1) \quad ax^2 \pm bx = \pm c.$$

$$(1) \times 4a = (2) \quad 4a^2x^2 \pm 4abx = \pm 4ac.$$

$$(3) \quad 4a^2x^2 \pm 4abx + b^2 = b^2 \pm 4ac.$$

$$\sqrt{(3)} = (4) \quad 2ax \pm b = \pm \sqrt{b^2 \pm 4ac}.$$

$$\therefore \quad (5) \quad x = \frac{\mp b \pm \sqrt{b^2 \pm 4ac}}{2a}.$$

282. Rule.

1. *Reduce the equation to one of the four forms indicated by the general equation* $ax^2 \pm bx = \pm c$.

2. *Write the unknown quantity equal to the co-efficient of the first power of the unknown quantity, with its sign changed plus or minus the square root of the sum of the squarc of the co-efficient of the first power of the unknown quantity, and four times the co-efficient of the second power of the unknown quantity into the second member, and all divided by twice the co-efficient of the second power of the unknown quantity.*

283. Examples.

1. Given $ax^2 + bx = c$, to find x.

$$Ans. \quad x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

2. Given $ax^2 - bx = c$, to find x.

$$Ans. \quad x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}.$$

3. Given $ax^2 + bx = -c$, to find x.

$$Ans.\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

4. Given $ax^2 - bx = -c$, to find x.

$$Ans.\ x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

5. Given $3x^2 + 7x = 110$, to find x.

Ans. $x = 5$ or $-7\frac{1}{3}$.

6. Given $5x^2 - 3x = 224$, to find x.

Ans. $x = 7$ or $-6\frac{2}{5}$.

7. Given $3x^2 + 25x = -8$, to find x.

Ans. $x = -\frac{1}{3}$ or -8.

8. Given $x^2 - 11x = -18$, to find x.

Ans. $x = 9$ or 2.

9. Given $(a-b)x^2 - (a+b)x = -\frac{ab}{a-b}$, to find x.

$$Ans.\ x = \frac{a}{a-b} \text{ or } \frac{b}{a-b}.$$

10. Given $x(x-1)d = 2s - 2ax$, to find x.

$$Ans.\ x = \frac{d - 2a \pm \sqrt{(d-2a)^2 + 8ds}}{2d}.$$

284. Special Formula.

If the co-efficient of the first power of the unknown quantity is even, the equation will take the form

$$(1) \quad ax^2 \pm 2bx = \pm c.$$

$$(1) \times a = (2) \quad a^2x^2 \pm 2abx = \pm ac.$$

$$(3) \quad a^2x^2 \pm 2abx + b^2 = b^2 \pm ac.$$

$$\sqrt{(3)} = (4) \quad ax \pm b = \pm\sqrt{b^2 \pm ac}.$$

$$\therefore \quad (5) \quad x = \frac{\mp b \pm \sqrt{b^2 \pm ac}}{a}.$$

285. Rule.

1. *Reduce the equation to one of the four forms indicated by the general equation* $ax^2 \pm 2bx = \pm c$.

2. *Write the unknown quantity equal to half the co-efficient of the first power of the unknown quantity with its sign changed, plus or minus the square root of the sum of the square of half the co-efficient of the first power of the unknown quantity, and the co-efficient of the second power of the unknown quantity into the second member, and all divided by the co-efficient of the second power of the unknown quantity.*

286. Examples.

1. Given $ax^2 + 2bx = c$, to find x.

Ans. $x = \dfrac{-b \pm \sqrt{b^2 + ac}}{a}$.

2. Given $ax^2 - 2bx = c$, to find x.

Ans. $x = \dfrac{b \pm \sqrt{b^2 + ac}}{a}$.

3. Given $ax^2 + 2bx = -c$, to find x.

Ans. $x = \dfrac{-b \pm \sqrt{b^2 - ac}}{a}$.

4. Given $ax^2 - 2bx = -c$, to find x.

Ans. $x = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$.

5. Given $3x^2 + 8x = 115$, to find x.

Ans. $x = 5$ or $-7\frac{2}{3}$.

6. Given $5x^2 - 12x = 224$, to find x.

Ans. $x = 8$ or $-5\frac{3}{5}$.

7. Given $3x^2 + 48x = -45$, to find x.

Ans. $x = -1$ or -15.

8. Given $5x^2 - 46x = -9$, to find x.

Ans. $x = 9$ or $\frac{1}{5}$.

9. Given $(a+b)x^2 - 2(a-b)x = \frac{4ab}{a+b}$. to find x.

Ans. $x = \frac{2a}{a+b}$ or $-\frac{2b}{a+b}$.

10. Given $(p-q)x^2 - 2p\sqrt{q}\,x = -pq$, to find x.

Ans. $x = \frac{p\sqrt{q} \pm q\sqrt{p}}{p-q}$.

287. Problems.

1. What number is that which being subtracted from 10 will leave a remainder equal to 25 divided by that number?

Ans. 5.

2. What are those two numbers whose difference is 15, and the cube of the less is equal to 5 times their product divided by 4? *Ans.* 5 and 20.

3. A drover bought cattle for $1350, and sold all but 20 for $1000, and gained $10 a head on those sold. How many did he buy? *Ans.* 45.

4. Divide 48 into two such parts that their product may be equal to 70 times their difference. *Ans.* 20 and 28.

5. Divide m into two such parts that their product may be equal to n times their difference.

$$Ans.\ \frac{m-2n\pm\sqrt{m^2+4n^2}}{2}\ \text{and}\ \frac{m+2n\mp\sqrt{m^2+4n^2}}{2}.$$

6. A merchant bought a certain number of pieces of cloth for \$375, and sold it at \$18 a piece, and gained by the trade 5 times the cost of one piece. How many pieces did he buy? *Ans.* 25.

7. A certain company wished to raise \$10000 stock in equal shares; but after 5 more men joined the company the shares were \$100 less. How many men were in the company at first? *Ans.* 20.

8. The base of a right-angled triangle + 5 is equal to the hypotenuse, and the perpendicular + 10 is equal to the hypotenuse. What are the sides? *Ans.* 25, 20, 15.

288. Formulas for Special Equations.

1. When the co-efficient of the first power of the unknown quantity is even, and the co-efficient of the second power is greater than unity, the equation is

$$ax^2 \pm 2bx = \pm c. \quad \therefore \quad x = \frac{\mp b \pm \sqrt{b^2 \pm ac}}{a}.$$

2. When the co-efficient of the first power of the unknown quantity is even, and the co-efficient of the second power is unity, the equation is

$$x^2 \pm 2bx = \pm c. \quad \therefore \quad x = \mp b \pm \sqrt{b^2 + c}.$$

3. When the co-efficient of the first power of the unknown quantity is odd, and the co-efficient of the second power is greater than unity, the equation is

$$ax^2 \pm bx = \pm c. \quad \therefore \quad x = \frac{\mp b \pm \sqrt{b^2 \pm 4ac}}{2a}.$$

4. When the co-efficient of the first power of the unknown quantity is odd, and the co-efficient of the second power is unity, the equation is

$$x^2 \pm bx = \pm c. \quad \therefore \quad x = \frac{\mp b \pm \sqrt{b^2 \pm 4c}}{2}.$$

289. Examples.

1. Given $3x^2 + 4x = 95$, to find x. *Ans.* 5 or $-\frac{19}{3}$.

2. Given $x^2 - 6x = 16$, to find x. *Ans.* 8 or -2.

3. Given $3x^2 + 17x = -10$, to find x.
Ans. $-\frac{2}{3}$ or -5.

4. Given $x^2 - 9x = -14$, to find x. *Ans.* 7 or 2.

290. Analogous Forms.

1. Given $x^{2n} + 2px^n = q$, to find x.

Since x^{2n} is the square of x^n, we have

$$x^n = -p \pm \sqrt{q + p^2}. \quad \therefore \quad x = \sqrt[n]{-p \pm \sqrt{q + p^2}}.$$

2. (1) $3x - 5\sqrt{x} = 12.$

$$\therefore \quad (2) \quad \sqrt{x} = \frac{5 \pm \sqrt{25 + 144}}{6}.$$

Or, (3) $\sqrt{x} = \frac{5 \pm 13}{6} = 3$ or $-1\frac{1}{3}$.

$(3)^2 = (4)$ $x = 9$ or $1\frac{7}{9}$.

3. (1) $x^{\frac{3}{2}} + 8x^{\frac{3}{4}} = 128$.

$\therefore$ (2) $x^{\frac{3}{4}} = -4 \pm \sqrt{16 + 128}$.

Or, (3) $x^{\frac{3}{4}} = -4 \pm 12 = 8$ or -16.

$\sqrt[3]{(3)} = (4)$ $x^{\frac{1}{4}} = 2$ or $(-16)^{\frac{1}{3}}$.

$(4)^4 = (5)$ $x = 16$ or $(-16)^{\frac{4}{3}}$.

4. (1) $(3x^2 - 8x)^2 - 6(3x^2 - 8x) = 7735$.

$\therefore$ (2) $3x^2 - 8x = 3 \pm \sqrt{9 + 7735}$.

Or, (3) $3x^2 - 8x = 3 \pm 88 = 91$ or -85.

1*st*. Let (4) $3x^2 - 8x = 91$.

$\therefore$ (5) $x = \frac{4 \pm \sqrt{16 + 273}}{3}$.

Or, (6) $x = \frac{4 \pm 17}{3} = 7$ or $-4\frac{1}{3}$.

2*d*. Let (7) $3x^2 - 8x = -85$.

$\therefore$ (8) $x = \frac{4 \pm \sqrt{16 - 255}}{3}$.

Or, (9) $x = \frac{4 \pm \sqrt{-239}}{3}$.

5. (1) $3(2x^2-7x)-5\sqrt{2x^2-7x+10}=20.$

(2) $3(2x^2-7x+10)-5\sqrt{2x^2-7x+10}=50.$

(3) $\sqrt{2x^2-7x+10}=\dfrac{5\pm\sqrt{25+600}}{6}.$

(4) $\sqrt{2x^2-7x+10}=\dfrac{5\pm25}{6}=5$ or $-\dfrac{10}{3}.$

$(4)^2=$ (5) $2x^2-7x+10=25$ or $\dfrac{100}{9}.$

1*st*. Let (6) $2x^2-7x=15.$

$\therefore$ (7) $x=\dfrac{7\pm\sqrt{49+120}}{4}.$

Or, (8) $x=\dfrac{7\pm13}{4}=5$ or $-\dfrac{3}{2}.$

2*d*. Let (9) $2x^2-7x=\dfrac{10}{9}.$

Or, (10) $18x^2-63x=10.$

$\therefore$ (11) $x=\dfrac{63\pm\sqrt{3969+720}}{36}.$

Or, (12) $x=\dfrac{63\pm3\sqrt{521}}{36}=\dfrac{21\pm\sqrt{521}}{12}.$

6. (1) $x^3-4x^2+6x=55.$

(2) $x^4-4x^3+6x^2-55x=0.$

(3) $(x^2-2x)^2+2x^2-55x=0.$

To make the first member a perfect square, since its first term is $(x^2-2x)^2$, such a quantity must be added as will make its second term of the form $n(x^2-2x)$, and its third term $\left(\frac{n}{2}\right)^2$. The same quantity must be added to the second member, which must be a perfect square whose first term is either x^2, $4x^2$, $9x^2$..., which may be tried in succession till the proper one is found. By a few trials, (3) may be put under the form

$$(4)\quad (x^2-2x)^2+11(x^2-2x)+\left(\frac{11}{2}\right)^2=9x^2+33x+\left(\frac{11}{2}\right)^2.$$

$$\sqrt{(4)}=(5)\quad x^2-2x+\frac{11}{2}=\pm\left(3x+\frac{11}{2}\right).$$

1*st.* Let (6) $x^2-5x=0$. $\therefore$ $x=5$ or 0.

2*d.* Let (7) $x^2+x=-11$. $\therefore$ $x=\dfrac{-1\pm\sqrt{-43}}{2}$.

291. Examples.

Find the values of x in the following examples.

1. $x^4-12x^2=64$. *Ans.* ± 4 or $\pm 2\sqrt{-1}$.

2. $x^6-3x^3=40$. *Ans.* 2 or $\sqrt[3]{-5}$.

3. $5x+24\sqrt{x}=176$. *Ans.* 16 or $77\frac{11}{25}$.

4. $x^3+5x^{\frac{3}{2}}=864$. *Ans.* 9 or $(-32)^{\frac{2}{3}}$.

5. $x^{\frac{2}{n}}-x^{\frac{1}{n}}=2$. *Ans.* 2^n or $(-1)^n$.

6. $x+5-\sqrt{x+5}=6$. *Ans.* 4 or -1.

7. $(3x^2 - 10x + 5)^2 - 8(3x^2 - 10x + 5) = -12.$

Ans. $\dfrac{5 \pm \sqrt{28}}{3}$, 3 or $\frac{1}{3}$.

8. $(4x^2 - 3x)^2 - 8(4x^2 - 3x - 2) = 36.$

Ans. 2, $-\frac{5}{4}$, $\dfrac{3 \pm \sqrt{-23}}{8}$.

9. $(3x - 5\sqrt{x} + 1)^2 - 7(3x - 5\sqrt{x}) = -5.$

Ans. 4, $\frac{1}{9}$, $\dfrac{43 \pm 5\sqrt{61}}{18}$.

10. $x^3 - 8x^2 - 28x + 80 = 0.$ *Ans.* 10, -4, 2.

11. $x^3 - 16x^2 + 20x + 112 = 0.$ *Ans.* 14, -2, 4.

12. $x^4 - 14x^3 + 31x^2 - 402x = 448.$

Ans. 14, -1, $\dfrac{1 \pm \sqrt{-127}}{2}$.

RECURRING EQUATIONS.

292. Definition.

A recurring equation is an equation in which the co-efficients taken in a direct or reverse order are the same numerically. Thus,

$$ax^3 - bx^2 + bx - a = 0.$$

293. Propositions.

1. *If the corresponding co-efficients of a recurring equation of an odd degree have like signs, one root is* -1, *and the equation*

is divisible by the unknown quantity $+1$; *but if these co-efficients have unlike signs, one root is* $+1$, *and the equation is divisible by the unknown quantity* -1.

Thus, -1 is a root of $ax^3+bx^2+bx+a=0$,

since -1 substituted gives $-a+b-b+a=0$.

Also, $+1$ is a root of $ax^3-bx^2+bx-a=0$,

since $+1$ substituted gives $a-b+b-a=0$.

2. *If the corresponding co-efficients of a recurring equation of an even degree, whose middle term is wanting, have unlike signs, one root is* $+1$ *and another root is* -1, *and the equation is divisible by the square of the unknown quantity* -1.

Thus, $+1$ and -1 are roots of $ax^4-bx^3+bx-a=0$,

since $+1$ substituted gives $a-b+b-a=0$,

and -1 substituted gives $a+b-b-a=0$.

294. Examples.

1. (1) $3x^3+5x^2+5x+3=0$.

(2) $(x+1)(3x^2+2x+3)=0$. Prop. 1.

$$\text{Hence,}\begin{cases}\text{either } x+1=0, & \therefore\ x=-1,\\ \text{or}\quad 3x^2+2x+3=0, & \therefore\ x=\dfrac{-1\pm\sqrt{-8}}{3}.\end{cases}$$

2. (1) $5x^3-7x^2+7x-5=0$.

(2) $(x-1)(5x^2-2x+5)=0$. Prop. 1.

$$\text{Hence,}\begin{cases}\text{either } x-1=0, & \therefore\ x=1,\\ \text{or}\quad 5x^2-2x+5=0, & \therefore\ x=\dfrac{1\pm2\sqrt{-6}}{5}.\end{cases}$$

3. (1) $2x^4 - 5x^3 + 5x - 2 = 0.$

(2) $(x^2 - 1)(2x^2 - 5x + 2) = 0.$ Prop. 2.

$$\text{Hence, } \begin{cases} \text{either } x^2 - 1 = 0, & \therefore\ x = \pm 1, \\ \text{or } \quad 2x^2 - 5x + 2 = 0, & \therefore\ x = 2 \text{ or } \tfrac{1}{2}. \end{cases}$$

4. (1) $ax^4 + bx^3 + cx^2 + bx + a = 0.$

$(1) \div x^2 = (2)$ $\quad ax^2 + bx + c + \dfrac{b}{x} + \dfrac{a}{x^2} = 0.$

(3) $ax^2 + 2a + \dfrac{a}{x^2} + bx + \dfrac{b}{x} + c - 2a = 0.$

(4) $a\left(x + \dfrac{1}{x}\right)^2 + b\left(x + \dfrac{1}{x}\right) = 2a - c.$

$\therefore$ (5) $x + \dfrac{1}{x} = \dfrac{-b \pm \sqrt{b^2 + 8a^2 - 4ac}}{2a}.$

Or, (6) $2ax^2 + (b \mp \sqrt{b^2 + 8a^2 - 4ac})\,x = -2a.$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 + 8a^2 - 4ac} \pm \sqrt{2b^2 - 8a^2 - 4ac \mp 2b\sqrt{b^2 + 8a^2 - 4ac}}}{4a}.$$

5. (1) $2x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 2 = 0.$

(2) $(x - 1)(2x^4 - x^3 + 3x^2 - x + 2) = 0.$

(3) $x - 1 = 0, \quad \therefore\ x = 1.$

(4) $2x^4 - x^3 + 3x^2 - x + 2 = 0.$

$(4) \div x^2 = (5)$ $\quad 2x^2 - x + 3 - \dfrac{1}{x} + \dfrac{2}{x^2} = 0.$

(6) $2x^2+4+\frac{2}{x^2}-x-\frac{1}{x}=1.$

(7) $2\left(x+\frac{1}{x}\right)^2-\left(x+\frac{1}{x}\right)=1.$

(8) $x+\frac{1}{x}=\frac{1\pm 3}{4}=1 \text{ or } -\frac{1}{2}.$

(9) $x^2-x=-1, \quad \therefore \quad x=\frac{1\pm\sqrt{-3}}{2}.$

(10) $2x^2+x=-2, \quad \therefore \quad x=\frac{-1\pm\sqrt{-15}}{4}.$

6. (1) $x^6-2x^5+3x^4-3x^2+2x-1=0.$

(2) $(x^2-1)(x^4-2x^3+4x^2-2x+1)=0.$

(3) $x^2-1=0, \quad \therefore \quad x=\pm 1.$

(4) $x^4-2x^3+4x^2-2x+1=0.$

(4) $\div x^2=$ (5) $x^2-2x+4-\frac{2}{x}+\frac{1}{x^2}=0.$

(6) $x^2+2+\frac{1}{x^2}-2x-\frac{2}{x}=-2.$

(7) $\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)=-2.$

(8) $x+\frac{1}{x}=1\pm\sqrt{-1}.$

(9) $x^2-(1\pm\sqrt{-1})\,x=-1.$

(10) $x=\frac{1\pm\sqrt{-1}\pm\sqrt{-4\pm 2\sqrt{-1}}}{2}.$

7. $4x^3 + 5x^2 + 5x + 4 = 0.$

Ans. $-1, \dfrac{-1 \pm \sqrt{-63}}{8}.$

8. $5x^3 - 12x^2 + 12x - 5 = 0.$

Ans. $1, \dfrac{7 \pm \sqrt{-51}}{10}.$

9. $6x^4 - 7x^3 + 7x - 6 = 0.$

Ans. $\pm 1, \dfrac{7 \pm \sqrt{-95}}{12}.$

10. $3x^5 - 4x^4 + 5x^3 - 5x^2 + 4x - 3 = 0.$

Ans. $1, \dfrac{1 \pm \sqrt{-3}}{2}, \dfrac{-1 \pm \sqrt{-8}}{3}.$

11. $5x^5 + 6x^4 + 7x^3 + 7x^2 + 6x + 5 = 0.$

Ans. $-1, \dfrac{2 \pm \sqrt{-21}}{5}, \dfrac{-1 \pm \sqrt{-3}}{2}.$

12. $2x^6 - 3x^5 + 4x^4 - 4x^2 + 3x - 2 = 0.$

Ans. $\pm 1, \dfrac{3 \pm \sqrt{-7} \pm \sqrt{-62 \pm 6\sqrt{-7}}}{8}.$

BINOMIAL EQUATIONS.

295. Definition.

A binomial equation is an equation of two terms of the form $x^n = a$.

296. Examples.

1. (1) $x^3 = a^3.$

(2) $x^3 - a^3 = 0.$

(3) $(x - a)(x^2 + ax + a^2) = 0.$

(4) $x - a = 0, \quad \therefore \quad x = a.$

(5) $x^2 + ax + a^2 = 0, \quad \therefore \quad x = \dfrac{-a \pm a\sqrt{-3}}{2}.$

2. (1) $x^4 - a^4 = 0.$

(2) $(x^2 + a^2)(x + a)(x - a) = 0.$

(3) $x - a = 0, \quad \therefore \quad x = a.$

(4) $x + a = 0. \quad \therefore \quad x = -a.$

(5) $x^2 + a^2 = 0, \quad \therefore \quad x = \pm a\sqrt{-1}.$

3. (1) $x^5 - a^5 = 0.$

(2) Let $x = ay$, then $x^5 = a^5y^5$.

$\therefore$ (3) $a^5y^5 - a^5 = 0.$

$(3) \div a^5 =$ (4) $y^5 - 1 = 0.$

(5) $(y - 1)(y^4 + y^3 + y^2 + y + 1) = 0.$

(6) $y - 1 = 0, \quad \therefore \quad y = 1.$

(7) $y^4 + y^3 + y^2 + y + 1 = 0.$

$(7) \div y^2 =$ (8) $y^2 + y + 1 + \dfrac{1}{y} + \dfrac{1}{y^2} = 0.$

(9) $y^2 + 2 + \dfrac{1}{y^2} + y + \dfrac{1}{y} = 1.$

(10) $\left(y + \dfrac{1}{y}\right)^2 + \left(y + \dfrac{1}{y}\right) = 1.$

(11) $y + \dfrac{1}{y} = \dfrac{-1 \pm \sqrt{5}}{2}.$

(12) $2y^2 + (1 \mp \sqrt{5})\, y = -2.$

(13) $y = \dfrac{-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}}{4}.$

(14) $x = a,\ a\left(\dfrac{-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}}{4}\right).$

4. (1) $x^6 - a^6 = 0.$

(2) $(x^3 + a^3)(x^3 - a^3) = 0.$

(3) $(x+a)(x^2 - ax + a^2)(x-a)(x^2 + ax + a^2) = 0.$

(4) $x + a = 0, \quad \therefore \quad x = -a.$

(5) $x^2 - ax + a^2 = 0, \quad \therefore \quad x = \dfrac{a \pm a\sqrt{-3}}{2}.$

(6) $x - a = 0, \quad \therefore \quad x = a.$

(7) $x^2 + ax + a^2 = 0, \quad \therefore \quad x = \dfrac{-a \pm a\sqrt{-3}}{2}.$

5. $x^3 = 8.$ *Ans.* $x = 2, \ -1 \pm \sqrt{-3}.$

6. $x^4 = 81.$ *Ans.* $\pm 3, \ \pm 3\sqrt{-1}.$

7. $x^5 = 32.$

Ans. $2, \ \dfrac{-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}}{2}.$

8. $x^6 = 15625.$ *Ans.* $\pm 5, \ \dfrac{\pm 5 \pm 5\sqrt{-3}}{2}.$

9. $x^8 = a^8.$

Ans. $\pm a, \ \pm a\sqrt{-1}, \ \pm a\sqrt[4]{-1}, \ \pm a\sqrt{-\sqrt{-1}}.$

10. $x^{10} = 1024.$

Ans. $2, \ \dfrac{-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}}{2},$

$-2, \ \dfrac{1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}}}{2}.$

298. Examples involving Radicals.

1. (1) $\sqrt{a^2 + x^2} - a = b.$

(2) $\sqrt{a^2 + x^2} = a + b.$

$(2)^2 = (3)$ $a^2 + x^2 = a^2 + 2ab + b^2.$

(4) $x^2 = 2ab + b^2.$

$\sqrt{(4)} = (5)$ $x = \pm \sqrt{2ab + b^2}.$

2. (1) $\dfrac{m + \sqrt{m^2 - x^2}}{m - \sqrt{m^2 - x^2}} = n.$

Multiplying both terms of the fraction by the numerator, we have

(2) $\dfrac{(m + \sqrt{m^2 - x^2})^2}{x^2} = n.$

$\sqrt{(2)} = (3)$ $\dfrac{m + \sqrt{m^2 - x^2}}{x} = \pm \sqrt{n}.$

Clearing of fractions and transposing, we have

(4) $\sqrt{m^2 - x^2} = \pm x\sqrt{n} - m.$

$(4)^2 = (5)$ $m^2 - x^2 = nx^2 \mp 2mx\sqrt{n} + m^2.$

(6) $(1 + n)x = \pm 2m\sqrt{n}.$

$$\therefore \quad x = \pm \frac{2m\sqrt{n}}{1 + n}.$$

3. Given $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=b$, to find x.

Ans. $x=\pm\dfrac{a(b-1)}{2\sqrt{b}}$.

4. Given $\dfrac{\sqrt{a+x}-\sqrt{x}}{\sqrt{a+x}+\sqrt{x}}=b$, to find x.

Ans. $x=\dfrac{a(b-1)^2}{4b}$.

5. Given $\dfrac{\sqrt{x}-\sqrt{x-p}}{\sqrt{x}+\sqrt{x-p}}=\dfrac{pq^2}{x-p}$, to find x.

Ans. $x=\dfrac{p(1\mp q)^2}{1\mp 2q}$.

6. Given $\dfrac{x+p+\sqrt{x^2+2px}}{x+p}=q$, to find x.

Ans. $x=\dfrac{\pm p(1\mp\sqrt{2q-q^2})}{\sqrt{2q-q^2}}$.

GENERAL DISCUSSION OF QUADRATICS.

299. Form of the Equation.

$$x^2+2px=q.$$

The co-efficient of x is denoted by $2p$ to avoid fractions; but $2p$ is not necessarily even, since p may be a fraction. In this equation, $2p$ and q may be either $+$ or $-$.

300. Propositions.

1. *Every equation of the second degree has two roots.*

Solving the equation

$$x^2+2px=q,$$

we have

$$x=-p\pm\sqrt{q+p^2}.$$

Separating the values of x, and denoting them respectively by x' and x'', we have

$$x' = -p + \sqrt{q + p^2}.$$
$$x'' = -p - \sqrt{q + p^2}.$$

Let it be remembered that by the roots we mean the values of x, and not simply the radical parts.

2. *An equation of the second degree can not have more than two roots.*

For, if possible, let a, b, and c be three different roots of

$$x^2 + 2px = q.$$

Then will these roots verify the equation, and give

$$(1) \quad a^2 + 2pa = q.$$

$$(2) \quad b^2 + 2pb = q.$$

$$(3) \quad c^2 + 2pc = q.$$

$$(1) - (2) = (4) \quad a^2 - b^2 + 2p(a - b) = 0.$$

$$(1) - (3) = (5) \quad a^2 - c^2 + 2p(a - c) = 0.$$

$$(4) \div (a - b) = (6) \quad a + b + 2p = 0.$$

$$(5) \div (a - c) = (7) \quad a + c + 2p = 0.$$

$$(6) - (7) = (8) \quad b - c = 0, \quad \therefore \quad b = c.$$

That is, the supposed third root does not differ from one of the other roots.

3. *The first member of every equation of the second degree whose second member is 0, can be resolved into two factors, having the unknown quantity for the first term of each, and the two roots, respectively, with their signs changed, for the second terms.*

(1) $x^2 + 2px = q.$

(2) $x^2 + 2px + p^2 = q + p^2.$

(3) $(x + p)^2 - (q + p^2) = 0.$

(4) $(x + p - \sqrt{q + p^2})(x + p + \sqrt{q + p^2}) = 0.$

By comparing these factors with the roots of the equation, the truth of the proposition will be evident.

4. *The sum of the roots is equal to the co-efficient of* x *with its sign changed.*

We shall prove this proposition in connection with the following:

5. *The product of the roots is equal to the second member with its sign changed.*

Attributing the signs to $2p$ and q, we have the forms:

1. $x^2 + 2px = q.$

$$\therefore \left\{\begin{array}{l} x' = -p + \sqrt{q + p^2}. \\ x'' = -p - \sqrt{q + p^2}. \end{array}\right\} \therefore \left\{\begin{array}{l} x' + x'' = -2p. \\ x'x'' = -q. \end{array}\right.$$

2. $x^2 - 2px = q.$

$$\therefore \left\{\begin{array}{l} x' = \quad p + \sqrt{q + p^2}. \\ x'' = \quad p - \sqrt{q + p^2}. \end{array}\right\} \therefore \left\{\begin{array}{l} x' + x'' = 2p. \\ x'x'' = -q. \end{array}\right.$$

3. $x^2 + 2px = -q.$

$$\therefore \left\{\begin{array}{l} x' = -p + \sqrt{-q + p^2}. \\ x'' = -p - \sqrt{-q + p^2}. \end{array}\right\} \therefore \left\{\begin{array}{l} x' + x'' = -2p. \\ x'x'' = q. \end{array}\right.$$

4. $x^2 - 2px = -q.$

$$\therefore \left\{\begin{array}{l} x' = \quad p + \sqrt{-q + p^2}. \\ x'' = \quad p - \sqrt{-q + p^2}. \end{array}\right\} \therefore \left\{\begin{array}{l} x' + x'' = 2p. \\ x'x'' = q. \end{array}\right.$$

6. *In the first form, the roots have unlike signs, and the negative root is numerically the greater.*

By Prop. 5, the product of the roots is equal to the second member with its sign changed; but in the first form, the second member is plus; hence, the product of the roots is minus; therefore, the roots have unlike signs.

By Prop. 4, the sum of the roots is equal to the co-efficient of x with its sign changed; but in the first form, this co-efficient is plus; hence, the sum of the roots is minus; therefore, the greater numerically is minus.

7. *In the second form, the roots have unlike signs, and the positive root is the greater.*

Let the student prove this proposition as above.

8. *In the third form, the roots have like signs, and both are negative.*

By Prop. 5, the product of the roots is equal to the second member with its sign changed; but in the third form, the second member is minus; hence, the product of the roots is plus; therefore, the roots have like signs.

By Prop. 4, the sum of the roots is equal to the co-efficient of x with its sign changed; but in the third form, this co-efficient is plus; hence, the sum of the roots is minus; therefore, both roots are minus.

9. *In the fourth form, the roots have like signs, and both are positive.*

Let the student prove this proposition as above.

Let Props. 6, 7, 8, and 9 be verified by an inspection of the roots.

301. Suppositions and Consequences.

1. $p^2 > q$.

Then, in all the forms, the roots are real, unequal, and rational or irrational, as is seen by inspecting the roots.

2. $p^2 = q$.

Then, in the first and second forms, the roots are real, unequal, and irrational; for, in the first form, $x' = -p + p\sqrt{2}$, $x'' = -p - p\sqrt{2}$; in the second form, $x' = p + p\sqrt{2}$, $x'' = p - p\sqrt{2}$. In the third and fourth forms, the roots are real, equal, and rational; for, in the third form, $x' = -p$, $x'' = -p$; in the fourth form, $x' = p$, $x'' = p$.

3. $p^2 < q$.

Then, in the first and second forms, the roots are real, unequal, and rational or irrational. In the third and fourth forms, the roots are *imaginary*, which indicates an impossibility; for, since $x' + x'' = 2p$ and $x'x'' = q$, numerically, let $x' = p + n$, $x'' = p - n$; then $x'x'' = p^2 - n^2 = q$, or $p^2 = q + n^2$; $\therefore$ p^2 can not be less than q.

4. $q = 0$.

Then, in the first and third forms, $x' = 0$, $x'' = -2p$; and in the second and fourth forms, $x' = 2p$, $x'' = 0$. This ought to be the case; for, since $x'x'' = q = 0$, either $x' = 0$ or $x'' = 0$; but since $x' + x'' = 2p$, with its sign changed, and since either $x' = 0$ or $x'' = 0$, the other must be equal to $2p$ with its sign changed.

5. $p = 0$.

Then, in the first and second forms, $x' = +\sqrt{q}$, $x'' = -\sqrt{q}$; and in the third and fourth forms, $x' = +\sqrt{-q}$, $x'' = -\sqrt{-q}$.

The roots are equal with contrary signs, which also is evident from the supposition, $p=0$; then $x'+x''=2p=0$. Since the sum of the roots is 0, they cancel each other in adding; they must therefore be equal with contrary signs. Indeed, the supposition reduces the equation to $x^2=\pm q$, an incomplete quadratic; $\therefore\ x=\pm\sqrt{\pm q}$.

6. $p=0$ and $q=0$.

Then, in all the forms, the equation becomes $x^2=0$, and $x'=0$, $x''=0$.

7. Let us take the equation

$$ax^2+bx=c.$$

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

If $$a=0,\ x'=\frac{0}{0},\ x''=\frac{-2b}{0}=-\infty;$$

but multiplying both terms of the value of x', which is

$$\frac{-b+\sqrt{b^2+4ac}}{2a},$$

by

$$-b-\sqrt{b^2+4ac},$$

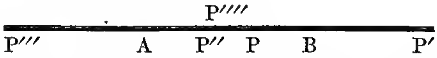
and reducing, we find

$$x'=\frac{2c}{b+\sqrt{b^2+4ac}}=\frac{c}{b},\ \text{if } a=0.$$

The infinite value of x'' indicates *impossibility*; for, since $a=0$, the equation becomes $bx=c$, an equation of the first degree, which can have but one root, $x=\frac{c}{b}$.

302. Problem of the Lights.

To find on the line joining two lights the points which are equally illuminated by those lights.



$$\text{Given}\begin{cases} a = \text{the intensity of light A at a unit's distance.} \\ b = \text{the intensity of light B at a unit's distance.} \\ d = \text{the distance between the lights.} \end{cases}$$

Let x = the distance from A to P, the point equally illuminated. Then $d - x$ = the distance from B to P.

It will be observed that we have assumed that there is a point, equally illuminated, between the lights, as is evidently true, and have made the notation to correspond to this supposition. Accordingly, the first value of x, denoted by x', ought to give this point. If there is another point, equally illuminated, not between the lights, as P′, the second value of x, denoted by x'', ought to give this point.

Since the intensity of the same light, at different distances, varies inversely as the square of the distance,

$$\frac{a}{x^2} = \text{the intensity of the light A at P.}$$

$$\frac{b}{(d-x)^2} = \text{the intensity of the light B at P.}$$

Since P is the point equally illuminated, we have

$$\frac{a}{x^2} = \frac{b}{(d-x)^2}, \quad \therefore \quad x = \frac{d\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}.$$

Separating these values of x, and denoting them by x' and x'', we have

$$x' = \frac{d\sqrt{a}}{\sqrt{a} + \sqrt{b}} \text{ and } x'' = \frac{d\sqrt{a}}{\sqrt{a} - \sqrt{b}}.$$

Let us now discuss these values under the following suppositions:

1. $a > b$ and $d > 0$.

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < 1 \text{ and } > \tfrac{1}{2}, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} < d \text{ and } > \tfrac{1}{2}d,$$

$$\therefore \quad x' = \text{AP}.$$

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > 1, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} > d, \quad \therefore \quad x'' = \text{AP}'.$$

Both points are nearer the weaker light; the first between the lights, the second not between the lights.

2. $a < b$ and $d > 0$.

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \tfrac{1}{2}, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \tfrac{1}{2}d, \quad \therefore \quad x' = \text{AP}''.$$

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} < 0, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} < 0, \quad \therefore \quad x'' = \text{AP}'''.$$

The negative value of x'' indicates a point at the left of A.

3. $a = b$ and $d > 0$.

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \tfrac{1}{2}, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \tfrac{1}{2}d, \quad \therefore \quad x' = \text{AP}''''.$$

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \infty, \quad \therefore \quad \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \infty, \quad \therefore \quad x'' = \infty.$$

The value ∞ indicates *impossibility*.

4. $a = b$ and $d = 0$.

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \tfrac{1}{2}, \text{ and } \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = 0, \quad \therefore \quad x' = 0.$$

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \infty, \text{ and } \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \frac{0}{0}, \quad \therefore \quad x'' = \frac{0}{0}.$$

The lights are together, since $d = 0$; and the value $x' = 0$ indicates that the point at which the lights are placed is equally illuminated; and the value $x'' = \frac{0}{0}$ indicates that any point of the line is equally illuminated by the lights.

5. $a > b$, or $a < b$ and $d = 0$.

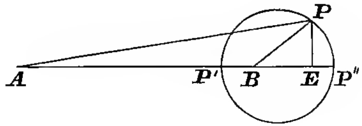
$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < 1, \qquad \text{and} \quad \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = 0, \quad \therefore \quad x' = 0.$$

$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > 1 \text{ or } < 0, \text{ and } \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} = 0, \quad \therefore \quad x'' = 0.$$

But shall we conclude that the point at which the lights are placed is equally illuminated by the lights which, by hypothesis, are of unequal intensities? The supposition $d = 0$ reduces the original equation $\frac{a}{x^2} = \frac{b}{(d-x)^2}$ to $\frac{a}{x^2} = \frac{b}{x^2}$, which is false under the hypothesis that $a > b$ or $a < b$, whatever be the value of x, even if $x = 0$; for then $\frac{a}{x^2}$ and $\frac{b}{x^2}$ become $\frac{a}{0}$ and $\frac{b}{0}$, which are unequal infinities.

303. Problem of the Lights Generalized.

To find all of the points equally illuminated by two lights.



1. $a > b$ and $d > 0$.

Finding, as before, the equally illuminated points on the line joining the lights, we have

$$x' = \frac{d\sqrt{a}}{\sqrt{a} + \sqrt{b}} = \text{AP}', \text{ and } x'' = \frac{d\sqrt{a}}{\sqrt{a} - \sqrt{b}} = \text{AP}''.$$

It is evident that there are other points, as the point P, not in the line joining the lights, which are equally illuminated. Let us find the *locus* of these points.

$$\frac{a}{\overline{\text{AP}}^2} = \frac{b}{\overline{\text{BP}}^2}, \quad \therefore \quad \frac{\overline{\text{BP}}^2}{\overline{\text{AP}}^2} = \frac{b}{a}, \quad \therefore \quad \frac{\text{BP}}{\text{AP}} = \frac{\sqrt{b}}{\sqrt{a}};$$

$$\therefore \quad \text{BP} : \text{AP} :: \sqrt{b} : \sqrt{a}.$$

Let $\qquad y\sqrt{b} = \text{BP};$

then $\qquad y\sqrt{a} = \text{AP}.$

Let PE be perpendicular to AE, and $n = \text{BE}$;

then $\quad ay^2 - (d+n)^2 = \overline{\text{PE}}^2$, and $by^2 - n^2 = \overline{\text{PE}}^2$.

$$\therefore \quad ay^2 - (d+n)^2 = by^2 - n^2; \quad \therefore \quad y^2 = \frac{d^2 + 2dn}{a - b}.$$

$$\therefore \quad \overline{\text{PE}}^2 = by^2 - n^2 = \frac{bd^2 + 2bdn}{a-b} - n^2.$$

$$EP' = AE - AP' = d + n - \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \frac{d\sqrt{b}}{\sqrt{a}+\sqrt{b}} + n.$$

$$EP'' = AP'' - AE = \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} - (d+n) = \frac{d\sqrt{b}}{\sqrt{a}-\sqrt{b}} - n.$$

$$\therefore\quad EP' \times EP'' = \left(\frac{d\sqrt{b}}{\sqrt{a}+\sqrt{b}} + n\right)\left(\frac{d\sqrt{b}}{\sqrt{a}-\sqrt{b}} - n\right).$$

By reducing, we find $EP' \times EP'' = \dfrac{bd^2 + 2bdn}{a-b} - n^2.$

$$\therefore\quad \overline{PE}^2 = EP' \times EP'',$$

which is a property of a circle.

Hence, the equally illuminated points are in the circumference of a circle around the weaker light, which, however, is not at the center. The diameter of this circle is

$$P'P'' = \frac{d\sqrt{a}}{\sqrt{a}-\sqrt{b}} - \frac{d\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \frac{2d\sqrt{ab}}{a-b},$$

which is the difference of the two values of x. As P may be any point in the circumference of this circle, it follows that all the points of this circumference are equally illuminated by the two lights.

Let this circumference be revolved about P′P″, as an axis. It will generate the surface of a sphere whose diameter is $\dfrac{2d\sqrt{ab}}{a-b}$; and since, in this revolution, the points in the circumference of the circle maintain their respective distances from the two lights, all the points of the circumference, throughout the revolution, will be equally illuminated. Hence, all the points in the surface of the generated sphere will be equally illuminated.

2. $a < b$ and $d > 0$.

In this case, all the points in the surface of a sphere about A, whose diameter is $\frac{2d\sqrt{ab}}{b-a}$, are equally illuminated.

3. $a = b$ and $d > 0$.

The diameter, $\frac{2d\sqrt{ab}}{a-b}$, of the sphere increases as a and b approach an equality; and, if $a = b$, the diameter becomes $\frac{2d\sqrt{ab}}{0} = \infty$, which indicates that the sphere is no longer possible. But, in this case, $y\sqrt{a} = y\sqrt{b}$, or AP $=$ BP, or the triangle APB becomes isosceles, and P is any point in the perpendicular to AB at its middle point.

Let this perpendicular be infinitely extended and then revolved about AB; it will generate a circle of infinite radius perpendicular to AB at its middle point, and all the points in the plane of this circle will be equally illuminated.

4. $a = b$ and $d = 0$.

On this supposition, the lights are of equal intensities and situated at the same point. Let P be any point in space whose distance from the lights is x, an indeterminate quantity. The intensity of the light A at this point is $\frac{a}{x^2}$, and of B, $\frac{b}{x^2}$; but $a = b$; $\therefore \frac{a}{x^2} = \frac{b}{x^2}$; hence, any point in space is equally illuminated by the lights.

5. $a > b$ or $a < b$ and $d = 0$.

On this supposition, the lights are of unequal intensities and are situated at the same point. Let P be any point

in space whose distance from the lights is x, an indeterminate quantity. The intensities of the lights A and B at this point are, respectively, $\frac{a}{x^2}$ and $\frac{b}{x^2}$; but $a > b$ or $a < b$; $\therefore \frac{a}{x^2} > \frac{b}{x^2}$, or $\frac{a}{x^2} < \frac{b}{x^2}$; hence, no point in space is equally illuminated by the lights.

TWO OR MORE UNKNOWN QUANTITIES.

304. Examples.

1. Given $\left\{\begin{array}{ll} (1) & x + y = s. \\ (2) & xy = p. \end{array}\right\}$ Required x and y.

$$(1)^2 = (3) \quad x^2 + 2xy + y^2 = s^2.$$

$$(3) - (2) \times 4 = (4) \quad x^2 - 2xy + y^2 = s^2 - 4p.$$

$$\sqrt{(4)} = (5) \quad x - y = \pm \sqrt{s^2 - 4p}.$$

$$\frac{(1) + (5)}{2} = (6) \quad x = \frac{s \pm \sqrt{s^2 - 4p}}{2}.$$

$$\frac{(1) - (5)}{2} = (7) \quad y = \frac{s \mp \sqrt{s^2 - 4p}}{2}.$$

2. Given $\left\{\begin{array}{ll} (1) & x + y = 10. \\ (2) & x^3 + y^3 = 280. \end{array}\right\}$ Required x and y.

$$(2) \div (1) = (3) \quad x^2 - xy + y^2 = 28.$$

$$(1)^2 = (4) \quad x^2 + 2xy + y^2 = 100.$$

$$(4) - (3) = (5) \quad 3xy = 72.$$

$$(5) \div 3 = (6) \quad xy = 24.$$

$$(3) - (6) = (7) \quad x^2 - 2xy + y^2 = 4.$$

$$\sqrt{(7)} = (8) \quad x - y = \pm 2.$$

$$\frac{(1)+(8)}{2} = (9) \quad x = 6 \text{ or } 4.$$

$$\frac{(1)-(8)}{2} = (10) \quad y = 4 \text{ or } 6.$$

3. Given $\left\{\begin{array}{ll}(1) & x+y=8. \\ (2) & x^4+y^4=706.\end{array}\right\}$ Required x and y.

$$(1)^4 = (3) \quad x^4+4x^3y+6x^2y^2+4xy^3+y^4=4096.$$

$$\frac{(2)+(3)}{2} = (4) \quad x^4+2x^3y+3x^2y^2+2xy^3+y^4=2401.$$

$$\sqrt{(4)} = (5) \quad x^2+xy+y^2=49.$$

$$(1)^2 = (6) \quad x^2+2xy+y^2=64.$$

$$(6)-(5) = (7) \quad xy=15.$$

$$(5)-(7)\times 3 = (8) \quad x^2-2xy+y^2=4.$$

$$\sqrt{(8)} = (9) \quad x-y=\pm 2.$$

$$\frac{(1)+(9)}{2} = (10) \quad x = 5 \text{ or } 3.$$

$$\frac{(1)-(9)}{2} = (11) \quad y = 3 \text{ or } 5.$$

4. Given $\left\{\begin{array}{ll}(1) & x+y=5. \\ (2) & x^5+y^5=275.\end{array}\right\}$ Required x and y.

$$(2)\div(1) = (3) \quad x^4-x^3y+x^2y^2-xy^3+y^4=55.$$

$$(1)^4 = (4) \quad x^4+4x^3y+6x^2y^2+4xy^3+y^4=625.$$

$$(4)-(3)=(5)\quad 5x^3y+5x^2y^2+5xy^3=570.$$

$$(5)\div 5xy=(6)\quad x^2+xy+y^2=\frac{114}{xy}.$$

$$(1)^2=(7)\quad x^2+2xy+y^2=25.$$

$$(7)-(6)=(8)\quad xy=25-\frac{114}{xy}.$$

$$(9)\quad x^2y^2-25xy=-114.$$

$$(10)\quad xy=\frac{25\pm\sqrt{625-456}}{2}=6.$$

$$(7)-(10)\times 4=(11)\quad x^2-2xy+y^2=1.$$

$$\sqrt{(11)}=(12)\quad x-y=\pm 1.$$

$$\frac{(1)+(12)}{2}=(13)\quad x=3 \text{ or } 2.$$

$$\frac{(1)-(12)}{2}=(14)\quad y=2 \text{ or } 3.$$

5. Given $\left\{\begin{array}{ll}(1) & x^{\frac{1}{5}}+y^{\frac{1}{4}}=5.\\ (2) & y^{\frac{3}{5}}+y^{\frac{3}{4}}=35.\end{array}\right\}$ Required x and y.

$$(3)\quad \text{Let } m=x^{\frac{1}{5}} \text{ and } n=y^{\frac{1}{4}}.$$

Then, $\left\{\begin{array}{ll}(4) & m+n=5.\\ (5) & m^3+n^3=35.\end{array}\right\}\therefore\left\{\begin{array}{l}m=3 \text{ or } 2.\\ n=2 \text{ or } 3.\end{array}\right.$

$\therefore\left\{\begin{array}{llll}(6) & x^{\frac{1}{5}}=3 \text{ or } 2. & \therefore & x=243 \text{ or } 32.\\ (7) & y^{\frac{1}{4}}=2 \text{ or } 3. & \therefore & y=16 \text{ or } 81.\end{array}\right.$

6. Given $\begin{cases} (1) & x^2 + xy + y^2 = 133. \\ (2) & x + \sqrt{xy} + y = 19. \end{cases}$

$$(1) \div (2) = (3) \quad x - \sqrt{xy} + y = 7.$$

$$\frac{(2) + (3)}{2} = (4) \quad x + y = 13.$$

$$\frac{(2) - (3)}{2} = (5) \quad \sqrt{xy} = 6.$$

$$(5)^2 = (6) \quad xy = 36.$$

$$(1) - (6) \times 3 = (7) \quad x^2 - 2xy + y^2 = 25.$$

$$\sqrt{(7)} = (8) \quad x - y = \pm 5.$$

$$\frac{(4) + (8)}{2} = (9) \quad x = 9 \text{ or } 4.$$

$$\frac{(4) - (8)}{2} = (10) \quad y = 4 \text{ or } 9.$$

7. Given $\left.\begin{cases} (1) & x^2 + 3xy + 2y^2 = 48. \\ (2) & 2x^2 + xy + y^2 = 44. \end{cases}\right\}$ Required x and y.

$$(3) \quad \text{Let } y = nx.$$

$$\therefore \begin{cases} (4) & (1 + 3n + 2n^2)\,x^2 = 48. \\ & \therefore \quad x^2 = \dfrac{48}{1 + 3n + 2n^2}. \\ (5) & (2 + n + n^2)\,x^2 = 44. \\ & \therefore \quad x^2 = \dfrac{44}{2 + n + n^2}. \end{cases}$$

$\therefore \quad (6) \quad \dfrac{48}{1+3n+2n^2} = \dfrac{44}{2+n+n^2}.$

$\therefore \quad (7) \quad 10n^2 + 21n = 13.$

$\therefore \quad (8) \quad n = \dfrac{-21 \pm \sqrt{441+520}}{20} = \frac{1}{2} \text{ or } -\frac{13}{5}.$

$\therefore \quad (9) \quad x^2 = \dfrac{44}{2+n+n^2} = 16 \text{ or } \frac{550}{77}.$

$\therefore \quad (10) \quad x = \pm 4 \text{ or } \pm\sqrt{\frac{550}{77}}.$

$\therefore \quad (11) \quad y = nx = \pm 2 \text{ or } \mp \frac{13}{5}\sqrt{\frac{550}{77}}.$

8. Given $\left\{\begin{array}{ll} (1) & x^2 + y^2 = 25. \\ (2) & x : \frac{12}{7} :: y : y - \frac{12}{7}. \end{array}\right\}$ Required x and y.

(2) gives $(3) \quad xy - \frac{12}{7}x = \frac{12}{7}y.$

$(3) \times 2 = (4) \quad 2xy - \frac{24}{7}(x+y) = 0.$

$(1) + (4) = (5) \quad (x+y)^2 - \frac{24}{7}(x+y) = 25.$

$\therefore \quad (6) \quad x + y = \frac{12}{7} \pm \sqrt{25 + \frac{144}{49}}.$

$\therefore \quad (7) \quad x + y = 7 \text{ or } -\frac{25}{7}.$

(7) in $(4) = (8) \quad 2xy = 24 \text{ or } -\frac{600}{49}.$

$(1) - (8) = (9) \quad (x-y)^2 = 1 \text{ or } \frac{1825}{49}.$

$\sqrt{(9)} = (10) \quad x - y = \pm 1 \text{ or } \pm \frac{5}{7}\sqrt{73}.$

$\dfrac{(7)+(10)}{2} = (11) \quad x = 4, 3 \text{ or } \dfrac{-25 \pm 5\sqrt{73}}{7}.$

$\dfrac{(7)-(10)}{2} = (12) \quad y = 3, 4 \text{ or } \dfrac{-25 \mp 5\sqrt{73}}{7}.$

9. $$\left\{\begin{array}{ll}(1) & \dfrac{xyz}{x+y}=a. \\ (2) & \dfrac{xyz}{x+z}=b. \\ (3) & \dfrac{xyz}{x+y}=c.\end{array}\right\} \therefore \left\{\begin{array}{ll}(4) & \dfrac{x+y}{xyz}=\dfrac{1}{a}. \\ (5) & \dfrac{x+z}{xyz}=\dfrac{1}{b}. \\ (6) & \dfrac{y+z}{xyz}=\dfrac{1}{c}.\end{array}\right\}$$

$$\frac{(4)+(5)+(6)}{2}=(7) \quad \frac{x+y+z}{xyz}=\frac{ab+ac+bc}{2abc}.$$

$$(7)-(4)=(8) \quad \frac{1}{xy}=\frac{ab+ac-bc}{2abc},$$

$$\therefore \quad (11) \quad xy=\frac{2abc}{ab+ac-bc}.$$

$$(7)-(5)=(9) \quad \frac{1}{xz}=\frac{ab+bc-ac}{2abc},$$

$$\therefore \quad (12) \quad xz=\frac{2abc}{ab+bc-ac}.$$

$$(7)-(6)=(10) \quad \frac{1}{yz}=\frac{ac+bc-ab}{2abc},$$

$$\therefore \quad (13) \quad yz=\frac{2abc}{ac+bc-ab}.$$

$$\sqrt{\frac{(11)\times(12)}{(13)}}=(14) \quad x=\sqrt{\frac{2abc\,(ac+bc-ab)}{(ab+ac-bc)\,(ab+bc-ac)}}.$$

$$\sqrt{\frac{(11)\times(13)}{(12)}}=(15) \quad y=\sqrt{\frac{2abc\,(ab+bc-ac)}{(ab+ac-bc)\,(ac+bc-ab)}}.$$

$$\sqrt{\frac{(12)\times(13)}{(11)}}=(16) \quad z=\sqrt{\frac{2abc\,(ab+ac-bc)}{(ab+bc-ac)\,(ac+bc-ab)}}.$$

10. Given $\left\{\begin{array}{ll}(1) & x^2+xy+y^2=37. \\ (2) & x^2+xz+z^2=28. \\ (3) & y^2+yz+z^2=19.\end{array}\right\}$ Required x, y, z.

$$\frac{[(1)-(2)]}{(y-z)}=(4) \quad x+y+z=\frac{9}{y-z}.$$

$$\frac{[(2)-(3)]}{(x-y)}=(5) \quad x+y+z=\frac{9}{x-y}.$$

$$\therefore \quad (6) \quad x-y=y-z.$$

$$\therefore \quad (7) \quad x=2y-z.$$

$$(7) \text{ in } (4)=(8) \quad 3y=\frac{9}{y-z}.$$

$$\therefore \quad (9) \quad z=\frac{y^2-3}{y}.$$

$$(9) \text{ in } (3)=(10) \quad y^2+y^2+\frac{(y^2-3)^2}{y^2}=22.$$

$$\therefore \quad (11) \quad 3y^4-28y^2=-9.$$

$$(12) \quad y^2=\frac{14\pm\sqrt{196-27}}{3}=9 \text{ or } \tfrac{1}{3}.$$

$$(13) \quad y=\pm 3 \text{ or } \pm\tfrac{1}{3}\sqrt{3}.$$

$$(14) \quad z=\frac{y^2-3}{y}=\pm 2 \text{ or } \mp\tfrac{8}{3}\sqrt{3}.$$

$$(15) \quad x=2y-z=\pm 4 \text{ or } \pm\tfrac{10}{3}\sqrt{3}.$$

11. $\left\{\begin{array}{l}x+y=12. \\ xy=35.\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=7 \text{ or } 5. \\ y=5 \text{ or } 7.\end{array}\right.$

12. $\left\{\begin{array}{l}x-y=7. \\ xy=30.\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=10 \text{ or } -3. \\ y=3 \text{ or } -10.\end{array}\right.$

13. $\begin{cases} x^2 + y^2 = 52. \\ x^2 - y^2 = 20. \end{cases}$ *Ans.* $\begin{cases} x = \pm 6. \\ y = \pm 4. \end{cases}$

14. $\begin{cases} x + y = 8. \\ x^2 + y^2 = 34. \end{cases}$ *Ans.* $\begin{cases} x = 5 \text{ or } 3. \\ y = 3 \text{ or } 5. \end{cases}$

15. $\begin{cases} x + y = 27. \\ x^3 + y^3 = 5103. \end{cases}$ *Ans.* $\begin{cases} x = 15 \text{ or } 12. \\ y = 12 \text{ or } 15. \end{cases}$

16. $\begin{cases} x + y = 7. \\ x^4 + y^4 = 337. \end{cases}$ *Ans.* $\begin{cases} x = 4 \text{ or } 3. \\ y = 3 \text{ or } 4. \end{cases}$

17. $\begin{cases} x + y = 8. \\ x^5 + y^5 = 3368. \end{cases}$ *Ans.* $\begin{cases} x = 5 \text{ or } 3. \\ y = 3 \text{ or } 5. \end{cases}$

18. $\begin{cases} x^{\frac{1}{4}} + y^{\frac{1}{4}} = 6. \\ x^{\frac{3}{4}} + y^{\frac{3}{4}} = 72. \end{cases}$ *Ans.* $\begin{cases} x = 256 \text{ or } 16. \\ y = 16 \text{ or } 256. \end{cases}$

19. $\begin{cases} x^2 + xy + y^2 = 481. \\ x + \sqrt{xy} + y = 37. \end{cases}$ *Ans.* $\begin{cases} x = 16 \text{ or } 9. \\ y = 9 \text{ or } 16. \end{cases}$

20. $\begin{cases} x^2 + y^2 + 3x - 3y = 40. \\ xy = 15. \end{cases}$

Ans. $\begin{cases} x = 5,\ -3,\ \dfrac{-5 \pm \sqrt{85}}{2}. \\ y = 3,\ -5,\ \dfrac{5 \pm \sqrt{85}}{2}. \end{cases}$

21. $\begin{cases} x + y + z = 12. \\ x + y - z = 2. \\ x^2 + y^2 = z^2. \end{cases}$ *Ans.* $\begin{cases} x = 4 \text{ or } 3. \\ y = 3 \text{ or } 4. \\ z = 5. \end{cases}$

22. $\left\{\begin{array}{l} 2x^2 + xy + 3y^2 = 64. \\ 2x^2 + 2xy - y^2 = 8. \end{array}\right\}$

$$Ans. \left\{\begin{array}{l} x = \pm 2 \text{ or } \pm \tfrac{22}{39}\sqrt{78}. \\ y = \pm 4 \text{ or } \mp \tfrac{14}{39}\sqrt{78}. \end{array}\right.$$

23. $\left\{\begin{array}{l} \dfrac{xyz}{x+y} = \tfrac{24}{7}. \\ \dfrac{xyz}{x+z} = 4. \\ \dfrac{xyz}{y+z} = \tfrac{24}{5}. \end{array}\right\}$ $\quad Ans. \left\{\begin{array}{l} x = \pm 4. \\ y = \pm 3. \\ z = \pm 2. \end{array}\right.$

24. $\left\{\begin{array}{l} x^2 + y^2 = h^2. \\ x : s :: y : y - s. \end{array}\right\}$

$$Ans. \left\{\begin{array}{l} x = \dfrac{s \pm \sqrt{h^2 + s^2} \pm \sqrt{h^2 - 2s\,(s \pm \sqrt{h^2 + s^2})}}{2}. \\ y = \dfrac{s \pm \sqrt{h^2 + s^2} \mp \sqrt{h^2 - 2s\,(s \mp \sqrt{h^2 + s^2})}}{2}. \end{array}\right.$$

25. $\left\{\begin{array}{l} wxy = 80. \\ wxz = 100. \\ wyz = 200. \\ xyz = 40. \end{array}\right\}$ $\quad Ans. \left\{\begin{array}{l} w = 10. \\ x = 2. \\ y = 4. \\ z = 5. \end{array}\right.$

305. Problems.

1. Find two numbers whose sum multiplied by the greater will produce 40, and whose difference multiplied by the less will produce 6.

Ans. ± 3 and ± 5, or $\pm\sqrt{2}$ and $\pm 4\sqrt{2}$.

2. The product of two numbers is 14, and the sum of their cubes is 351. Required the numbers.

Ans. 2 and 7.

3. The product of two numbers is 15, and the sum of their fourth powers is 706. Required the numbers.

Ans. 3 and 5.

4. Find three numbers whose sum is 12, the sum of whose squares is 50, and the product of the first and third plus 1 is equal to the square of the second.

Ans. 5, 4, 3.

5. There are three rectangles, the length of each is twice its width, the sum of their lengths is 48, the sum of their areas is 400, and the area of the second is equal to the sum of the areas of the first and third. Find the area of each. *Ans.* 72, 200, 128.

6. If the sum of the squares of two numbers plus their product be divided by their sum, the quotient will be 14; and if the sum of their squares minus their product be divided by their difference, the quotient will be 18. Required the numbers. *Ans.* 6 and 12.

RATIO.

306. Definitions.

1. Ratio is the relation of two quantities of the same kind, expressed by the quotient obtained by dividing the first by the second. A ratio is therefore an abstract number.

2. The **terms** of a ratio are the quantities compared. The first term is the antecedent, the second the consequent. The antecedent and consequent together form a couplet.

3. **The sign** of a ratio is the colon (:). A ratio is expressed by placing the sign between the terms. Thus, 4 : 7 expresses the ratio of 4 to 7, and is read the ratio of 4 to 7.

Let a denote the antecedent, c the consequent, and r the ratio. Then, by definition, we shall have

$$r = a : c = \frac{a}{c}.$$

307. Classification of Ratios.

1. *A simple ratio* is a ratio of integral terms.

2. *A complex ratio* is a ratio of fractional terms.

3. *A compound ratio* is the product of two or more ratios.

4. *A duplicate ratio* is the square of a given ratio.

5. *A triplicate ratio* is the cube of a given ratio.

6. *A subduplicate ratio* is the square root of a given ratio.

7. *A subtriplicate ratio* is the cube root of a given ratio.

REDUCTION OF RATIOS.

308. Case I.

To reduce simple ratios to their lowest terms.

$$a : c = \frac{a}{c} = \frac{a \div n}{c \div n} = (a \div n) : (c \div n).$$

Hence, divide both terms by their g. c. d.

309. Examples.

1. $120 : 440.$ *Ans.* $3 : 11.$

2. $a^2 - b^2 : a^2 - 2ab + b^2.$ *Ans.* $a + b : a - b.$

3. $x^2 - 7x + 12 : x^2 - 9x + 20.$ *Ans.* $x - 3 : x - 5.$

310. Case II.

To reduce complex ratios to simple ratios.

$$a : c = \frac{a}{c} = \frac{a \times n}{c \times n} = a \times n : c \times n.$$

Thus, $\frac{3}{4} : 5\frac{1}{3} = 9 : 64$, by multiplying both terms by 12, the l. c. m. of the denominators 4 and 3. Hence,

Multiply both terms by the l. c. m. of the denominators.

311. Examples.

1. $\frac{3}{4} : \frac{11}{6}.$ *Ans.* $9 : 22$

2. $a : b + \frac{c}{d}.$ *Ans.* $ad : bd + c.$

3. $\frac{a}{b} : \frac{c}{d}.$ *Ans.* $ad : bc.$

312. Case III.

To reduce compound ratios to simple ratios.

$$(a : b) \times (c : d) = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = ac : bd.$$

Hence, multiply the several ratios together.

313. Examples.

1. $(7:8)\times(8:14)$. *Ans.* $1:2$.

2. $(3\frac{1}{2}:6\frac{1}{3})\times(4\frac{3}{4}:5\frac{7}{8})$. *Ans.* $21:47$.

3. $\left(m+\dfrac{3mn+n^2}{m-n}:m+n\right)\times(m+n:m-n)$.
Ans. $(m+n)^2:(m-n)^2$.

314. General Problems.

1. Given a and c, required r. *Ans.* $r=\dfrac{a}{c}$.

2. Given c and r, required a. *Ans.* $a=c\times r$.

3. Given a and r, required c. *Ans.* $c=\dfrac{a}{r}$.

315. Examples.

1. Given $a=x^3-y^3$, $c=x-y$, required r.
Ans. $r=x^2+xy+y^2$.

2. Given $c=x^4-x^2y^2+y^4$, $r=x^2+y^2$, required a.
Ans. $a=x^6+y^6$.

3. Given $a=m^2-n^2$, $r=(m+n)(\sqrt{m}+\sqrt{n})$, required c.
Ans. $c=\sqrt{m}-\sqrt{n}$.

PROPORTION.

316. Definitions.

1. A proportion is an equality of ratios. Thus,

$$(1)\quad a:b::c:d$$

is a proportion, and is read a is to b as c is to d.

By definition of ratio and proportion, (1) is equivalent to

$$(2) \quad \frac{a}{b}=\frac{c}{d}.$$

2. **The terms** of a proportion are the quantities compared. The first and second terms form the first couplet; the third and fourth, the second couplet. The first and fourth terms are the extremes; the second and third, the means. The first and third are the antecedents; the second and fourth, the consequents.

The terms of each couplet, the antecedents, and the consequents are corresponding terms.

3. A proportion is taken by *alternation* when antecedent is compared with antecedent, and consequent with consequent. If $a : b :: c : d$, by alternation, $a : c :: b : d$.

4. A proportion is taken by *inversion* when the consequents are taken for antecedents, and the antecedents for consequents. If $a : b :: c : d$, by inversion, $b : a :: d : c$.

5. A proportion is taken by *composition* when the sum of the terms of each couplet is compared with either antecedent or consequent. If $a : b :: c : d$, by composition,

$$a+b : a :: c+d : c, \text{ or } a+b : b :: c+d : d.$$

6. A proportion is taken by *division* when the difference of the terms of each couplet is compared with either antecedent or consequent. If $a : b :: c : d$, by division,

$$a-b : a :: c-d : c, \text{ or } a-b : b :: c-d : d.$$

7. A proportion is taken by *composition and division* when the sum of the terms of each couplet is compared with their difference. If $a : b :: c : d$, by composition and division,

$$a+b : a-b :: c+d : c-d.$$

8. **Equimultiples** of two or more quantities are the products of those quantities by a common multiplier. Thus, ma and mb are equimultiples of a and b.

9. **Equi-submultiples** of two or more quantities are the quotients of those quantities by a common divisor. Thus, $\frac{a}{m}$ and $\frac{b}{m}$ are equi-submultiples of a and b.

317. Propositions.

1. *If the terms of one couplet are equal, the terms of the other couplet are also equal.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}.$

If $a = b$, then $\frac{a}{b} = 1$; hence, $\frac{c}{d} = 1$; $\therefore$ $c = d$.

2. *If the antecedents are equal, the consequents are also equal, and conversely.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}.$

If (3) $a = c,$

then (2) $\div$ (3) $=$ (4) $\frac{1}{b} = \frac{1}{d}.$

(4) $\times\, bd =$ (5) $d = b.$

If (6) $b = d,$

then (2) $\times$ (6) $=$ (7) $a = c.$

3. *If one antecedent is greater or less than its consequent, the other antecedent is also greater or less than its consequent.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}.$

If $a > b$, then $\frac{a}{b} > 1$; hence, $\frac{c}{d} > 1$; $\therefore\ c > d.$

If $a < b$, then $\frac{a}{b} < 1$; hence, $\frac{c}{d} < 1$; $\therefore\ c < d.$

4. *If one antecedent is greater or less than the other antecedent, its consequent is also greater or less than the other consequent.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}.$

If $a > c$, then $b > d$; otherwise, $\frac{a}{b} > \frac{c}{d}.$

If $a < c$, then $b < d$; otherwise, $\frac{a}{b} < \frac{c}{d}.$

5. *In every proportion, the product of the extremes is equal to the product of the means.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}.$

$$(2) \times bd = (3) \quad ad = bc.$$

6. *If the product of two quantities is equal to the product of two other quantities, the factors of either product may be made the extremes, and the factors of the other product the means of a proportion.*

If $\quad (1) \quad a \times d = b \times c,$

then $\quad (1) \div bd = (2) \quad \dfrac{a}{b} = \dfrac{c}{d}.$

$\therefore \quad (3) \quad a : b :: c : d.$

7. *If four quantities are in proportion, they will be in proportion by alternation.*

If $\quad (1) \quad a : b :: c : d,$

then $\quad (2) \quad \dfrac{a}{b} = \dfrac{c}{d}.$

$(2) \times \dfrac{b}{c} = (3) \quad \dfrac{a}{c} = \dfrac{b}{d}.$

$\therefore \quad (4) \quad a : c :: b : d.$

8. *If four quantities are in proportion, they will be in proportion by inversion.*

If $\quad (1) \quad a : b :: c : d,$

then $\quad (2) \quad \dfrac{a}{b} = \dfrac{c}{d}.$

Taking the reciprocal of each member of (2), we have

$(3) \quad \dfrac{b}{a} = \dfrac{d}{c}.$

$\therefore \quad (4) \quad b : a :: d : c.$

9. *If two proportions have a ratio of the one equal to a ratio of the other, the remaining terms are proportional.*

If $\quad (1) \quad a : b :: c : d,$

then $\quad (2) \quad \dfrac{a}{b} = \dfrac{c}{d}.$

If (3) $e : f :: g : h$,

then (4) $\dfrac{e}{f} = \dfrac{g}{h}$.

If (5) $a : b :: e : f$,

then (6) $\dfrac{a}{b} = \dfrac{e}{f}$.

$\therefore$ (7) $\dfrac{c}{d} = \dfrac{g}{h}$.

$\therefore$ (8) $c : d :: g : h$.

Cor. If the antecedents of two proportions are in proportion, the consequents are in proportion, and conversely. For, by alternation of (1) and (3), we have

(9) $a : c :: b : d$,

and (10) $e : g :: f : h$.

If (11) $a : c :: e : g$,

then (12) $b : d :: f : h$.

If (13) $b : d :: f : h$,

then (14) $a : c :: e : g$.

10. *If four quantities are in proportion, they will be in proportion by composition.*

If (1) $a : b :: c : d$,

then (2) $ad = bc$. Prop. 5.

(3) $ac = ac$.

(2) + (3) = (4) $a(c + d) = c(a + b)$.

$\therefore$ (5) $a + b : a :: c + d : c$. Prop. 6.

11. *If four quantities are in proportion, they will be in proportion by division.*

If (1) $a : b :: c : d$,

then (2) $ad = bc$. Prop. 5.

(3) $ac = ac$.

(3) — (2) = (4) $a(c - d) = c(a - b)$.

$\therefore$ (5) $a - b : a :: c - d : c$. Prop. 6.

12. *If four quantities are in proportion, they will be in proportion by composition and division.*

If (1) $a : b :: c : d$,

then (2) $a + b : a :: c + d : c$. Prop. 10.

And (3) $a - b : a :: c - d : c$. Prop. 11.

$\therefore$ (4) $a + b : c + d :: a - b : c - d$. Prop. 9, Cor.

$\therefore$ (5) $a + b : a - b :: c + d : c - d$. Prop. 7.

13. *Two quantities have the same ratio as their equimultiples or equi-submultiples.*

(1) $\frac{a}{b} = \frac{ma}{mb}$, $\therefore$ $a : b :: ma : mb$.

(2) $\frac{a}{b} = \frac{\frac{a}{m}}{\frac{b}{m}}$, $\therefore$ $a : b :: \frac{a}{m} : \frac{b}{m}$.

14. *If equimultiples or equi-submultiples be taken of all the terms of a proportion, or of any two corresponding terms, the resulting quantities will be in proportion.*

If (1) $a : b :: c : d,$

then (2) $\frac{a}{b} = \frac{c}{d}$.

Hence, (3) $\frac{ma}{mb} = \frac{mc}{md}$, $\therefore$ $ma : mb :: mc : md.$

(4) $\frac{\frac{a}{m}}{\frac{b}{m}} = \frac{\frac{c}{m}}{\frac{d}{m}}$, $\therefore$ $\frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}$.

(5) $\frac{ma}{mb} = \frac{c}{d}$, $\therefore$ $ma : mb :: c : d.$

(6) $\frac{\frac{a}{m}}{\frac{b}{m}} = \frac{c}{d}$, $\therefore$ $\frac{a}{m} : \frac{b}{m} :: c : d.$

(7) $\frac{a}{b} = \frac{mc}{md}$, $\therefore$ $a : b :: mc : md.$

(8) $\frac{a}{b} = \frac{\frac{c}{m}}{\frac{d}{m}}$, $\therefore$ $a : b :: \frac{c}{m} : \frac{d}{m}$.

(9) $\frac{ma}{b} = \frac{mc}{d}$, $\therefore$ $ma : b :: mc : d.$

(10) $\frac{\frac{a}{m}}{b} = \frac{\frac{c}{m}}{d}$, $\therefore$ $\frac{a}{m} : b :: \frac{c}{m} : d.$

(11) $\frac{a}{mb} = \frac{c}{md}$, $\therefore$ $a : mb :: c : md.$

(12) $\frac{a}{\frac{b}{m}} = \frac{c}{\frac{d}{m}}$, $\therefore$ $a : \frac{b}{m} :: c : \frac{d}{m}$.

(13) $\frac{ma}{mb} = \frac{nc}{nd}$, $\therefore$ $ma : mb :: nc : nd$.

(14) $\frac{\frac{a}{m}}{\frac{b}{m}} = \frac{\frac{c}{n}}{\frac{d}{n}}$, $\therefore$ $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$.

(15) $\frac{ma}{nb} = \frac{mc}{nb}$, $\therefore$ $ma : nb :: mc : nb$.

(16) $\frac{\frac{a}{m}}{\frac{b}{n}} = \frac{\frac{c}{m}}{\frac{b}{n}}$, $\therefore$ $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

15. *If two quantities be increased or diminished by quantities having the same ratio, the resulting quantities will have the same ratio.*

If (1) $a : b :: c : d$,

then (2) $ad = bc$.

(3) $ab = ab$.

(3) $\pm$ (2) = (4) $a(b \pm d) = b(a \pm c)$.

$\therefore$ (5) $a : b :: a \pm c : b \pm d$.

Cor. 1. If two quantities be increased or diminished by their equimultiples or equi-submultiples, the resulting quantities will have the same ratio.

Cor. 2. If the antecedents or consequents be increased or diminished by quantities having the same ratio, the results will have the same ratio as the consequents or antecedents.

16. *In a continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

(1) $a : b :: c : d :: e : f :: g : h.$

(2) $ab = ab.$

(3) $ad = bc.$

(4) $af = be.$

(5) $ah = bg.$

Taking the sum of (2), (3), (4), (5), ..., we have

(6) $a(b + d + f + h + \dots) = b(a + c + e + g + \dots).$

$\therefore$ (7) $a + c + c + g + \dots : b + d + f + h + \dots :: a : b.$

17. *The products of the corresponding terms of two or more proportions are in proportion.*

If (1) $a : b :: c : d$, then (4) $\dfrac{a}{b} = \dfrac{c}{d}.$

If (2) $e : f :: g : h$, then (5) $\dfrac{e}{f} = \dfrac{g}{h}.$

If (3) $i : j :: k : l$, then (6) $\dfrac{i}{j} = \dfrac{k}{l}.$

Taking the product of (4), (5), (6), ..., we have

(7) $\dfrac{aei\dots}{bfj\dots} = \dfrac{cgk\dots}{dhl\dots}.$

$\therefore$ (8) $aei\dots : bfj\dots :: cgk\dots : dhl\dots$

Cor. If $a = e = i = \dots$, $b = f = j = \dots$, $c = g = k = \dots$, $d = h = l = \dots$, then (8) becomes

(9) $a^n : b^n :: c^n : d^n.$

18. *The quotients of the corresponding terms of two proportions are in proportion.*

If (1) $a : b :: c : d$, then (3) $\frac{a}{b} = \frac{c}{d}$.

If (2) $e : f :: g : h$, then (4) $\frac{e}{f} = \frac{g}{h}$.

$$(3) \div (4) = (5) \quad \frac{\frac{a}{e}}{\frac{b}{f}} = \frac{\frac{c}{g}}{\frac{d}{h}}. \quad \therefore \quad (6) \quad \frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}.$$

19. *Like powers or roots of the terms of a proportion are in proportion.*

If (1) $a : b :: c : d$,

then (2) $\frac{a}{b} = \frac{c}{d}$.

$$(2)^n = (3) \quad \frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

$$\therefore \quad (4) \quad a^n : b^n :: c^n : d^n.$$

$$\sqrt[n]{(2)} = (5) \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}},$$

$$\therefore \quad (6) \quad \sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}.$$

318. Examples in Proportion.

1. Given $a^2 - b^2 : a + b :: (a - b)^2 : x$, required x.
Ans. $x = a - b$.

2. If $a : b :: b : c$, prove that $a : c :: a^2 : b^2$.

3. If $a : b :: b : c :: c : d$, prove that $a : d :: a^3 : b^3$.

4. What number added to each of the numbers 2, 4, 5, 8 will make the results proportional? *Ans.* 4.

5. What quantity added to each of the quantities m, n, p, q will make the results proportional?

$$Ans.\ \frac{np - mq}{m + q - n - p}.$$

6. If $a : b :: c : d$, prove that there is no quantity which added to each will make the results proportional.

7. Given $\begin{cases} (1) & xy = 48. \\ (2) & x^3 + y^3 : x^3 - y^3 :: 91 : 37. \end{cases}$ Required x and y.

Ans. $x = 8$, $y = 6$.

8. Given $\sqrt[3]{x + a} : \sqrt[3]{x - a} :: m : n$, required x.

$$Ans.\ x = \frac{a(m^3 + n^3)}{m^3 - n^3}.$$

9. Find two quantities whose sum, difference, and product are proportional to m, n, and p.

$$Ans.\ \frac{2p}{m - n} \text{ and } \frac{2p}{m + n}.$$

10. The product of two numbers is 15, and the cube of their sum is to the sum of their cubes as 64 is to 19. Required the numbers. *Ans.* 5 and 3.

VARIATION.

319. Definitions.

1. One quantity varies directly as another, when it increases or decreases in the same ratio as the other. Thus, the space described by a body in uniform motion varies directly as the time, which is thus expressed,

$$s \propto t.$$

2. One quantity varies inversely as another when it increases in the same ratio as the other decreases, or decreases in the same ratio as the other increases. Thus, in uniform motion, if the space is constant, the time varies inversely as the rate, which is thus expressed,

$$t \propto \frac{1}{r}.$$

3. One quantity varies as two others jointly, when the first increases or decreases in the same ratio as the product of the second and third. Thus, the space described by a body in motion varies as the rate and time jointly, which is thus expressed,

$$s \propto rt.$$

4. One quantity varies directly as a second quantity, and inversely as a third, when it increases or decreases in the same ratio as the quotient of the second by the third. Thus, if a body is in motion, the time varies directly as the space and inversely as the rate, which is thus expressed,

$$t \propto \frac{s}{r}.$$

320. Propositions.

1. *If one quantity varies as another, the first is equal to the second, multiplied by a constant.*

Let $x \propto y$, and let x' and y', x'' and y'', ..., be corresponding states of x and y; that is, if x becomes x', y becomes y', ... Then, by definition, we shall have

$$(1) \quad x : x' :: y : y'.$$

$$(2) \quad x : x'' :: y : y''.$$

$\therefore$ (3) $x' : y' :: x'' : y''$. Art. 317, Prop. 9, Cor.

Or, (4) $\dfrac{x'}{y'} = \dfrac{x''}{y''}$.

Therefore, the ratio of the corresponding states of x and y is constant. Let this constant be denoted by m; then,

$$(5) \quad m = \frac{x'}{y'} = \frac{x''}{y''}.$$

(1) gives (6) $x = \dfrac{x'}{y'}\, y$;

then (7) $x = my$.

2. *If one variable quantity is equal to another multiplied by a constant, the first varies as the second.*

Let x and y be two variable quantities, and x' and y' any corresponding states of x and y.

If (1) $x = my$, then (2) $x' = my'$.

$\therefore$ (3) $\dfrac{x}{y} = m$, and (4) $\dfrac{x'}{y'} = m$.

$\therefore$ (5) $\dfrac{x}{y} = \dfrac{x'}{y'}$; or, (6) $x : y :: x' : y'$;

or, (7) $x : x' :: y : y'$. $\therefore$ (8) $x \propto y$.

3. *If one quantity varies as a second, and the second as a third, the first varies as the third.*

If (1) $x \propto y$, then (2) $x = my$.

If (3) $y \propto z$, then (4) $y = nz$.

Substituting the value of y given in (4) in (2), we have

$$(5) \quad x = mnz.$$

But since m and n are constant, mn is constant, $\therefore$ $x \propto z$. Prop. 2.

4. *If each of two quantities varies as a third, their sum, their difference, or their mean proportional varies as the third.*

If $\quad (1) \quad x \propto z, \quad$ then $\quad (2) \quad x = mz.$

If $\quad (3) \quad y \propto z, \quad$ then $\quad (4) \quad y = nz.$

$$(2) \pm (4) = (5) \quad x \pm y = (m \pm n)\, z,$$

$$\therefore \quad (6) \quad x \pm y \propto z.$$

$$\sqrt{(2) \times (4)} = (7) \quad \sqrt{xy} = \sqrt{mn}\, z,$$

$$\therefore \quad (8) \quad \sqrt{xy} \propto z.$$

5. *If one quantity varies as two others jointly, either of the latter varies directly as the first, and inversely as the other.*

If $\quad (1) \quad x \propto yz, \quad$ then $\quad (2) \quad x = myz.$

Therefore, $\quad (3) \quad y = \frac{1}{m} \frac{x}{z}, \quad \therefore \quad (4) \quad y \propto \frac{x}{z},$

and $\quad (5) \quad z = \frac{1}{m} \frac{x}{y}, \quad \therefore \quad (6) \quad z \propto \frac{x}{y}.$

6. *If one quantity varies as another, it will vary as any multiple or submultiple of the other.*

If $\quad (1) \quad x \propto y, \quad$ then $\quad (2) \quad x = my.$

$$(2) = (3) \quad x = \frac{m}{n} ny, \quad \therefore \quad (4) \quad x \propto ny.$$

$$(2) = (5) \quad x = mn \frac{1}{n} y, \quad \therefore \quad (6) \quad x \propto \frac{1}{n} y.$$

7. *If one quantity varies as another, and each of them be multiplied or divided by any quantity, variable or invariable, the products or quotients will vary as each other.*

If (1) $x \propto y$, then (2) $x = my$.

Let z be any quantity, variable or invariable.

Then (3) $xz = myz$, $\therefore$ (4) $xz \propto yz$,

and (5) $\frac{x}{z} = m\frac{y}{z}$, $\therefore$ (6) $\frac{x}{z} \propto \frac{y}{z}$.

8. *The products of the corresponding members of two or more variations vary as each other.*

If (1) $p \propto q$, then (2) $p = aq$.

If (3) $r \propto s$, then (4) $r = bs$.

If (5) $t \propto u$, then (6) $t = cu$.

. . . .

Taking the product of (2), (4), (6), . . . , we have

(7) $prt \ldots = abc \ldots qsu \ldots$

$\therefore$ (8) $prt \ldots \propto qsu \ldots$

If $p = r = t = \ldots$, and $q = s = u = \ldots$, then (8) becomes

(9) $p^n \propto q^n$.

9. *The quotients of the corresponding members of two variations vary as each other.*

If (1) $u \propto x$, then (2) $u = mx$.

If (3) $y \propto z$, then (4) $y = nz$.

(2) $\div$ (4) $=$ (5) $\frac{u}{y} = \frac{m}{n}\frac{x}{z}$; $\therefore$ (6) $\frac{u}{y} \propto \frac{x}{z}$.

10. *Like powers or roots of the members of a variation vary as each other.*

If (1) $x \propto y$, then (2) $x = my$.

$(2)^n =$ (3) $x^n = m^n y^n$, $\therefore$ (4) $x^n \propto y^n$.

$\sqrt[n]{(2)} =$ (5) $\sqrt[n]{x} = \sqrt[n]{m}\sqrt[n]{y}$, $\therefore$ (6) $\sqrt[n]{x} \propto \sqrt[n]{y}$.

11. *If* x $\propto$ y *when* z *is constant, and as* z *when* y *is constant, then* x $\propto$ yz *when* y *and* z *are variables.*

Let us suppose that y and z vary in succession, and that in consequence of y becoming y', x becomes x', and that in consequence of z becoming z', x' becomes x''. Then,

(1) $x : x' :: y : y'$, $\therefore$ (3) $\dfrac{x}{x'} = \dfrac{y}{y'}$.

(2) $x' : x'' :: z : z'$, $\therefore$ (4) $\dfrac{x'}{x''} = \dfrac{z}{z'}$.

(3) $\times$ (4) $=$ (5) $\dfrac{x}{x''} = \dfrac{yz}{y'z'}$, or $x = \dfrac{x''}{y'z'}yz$, $\therefore$ $x \propto yz$.

12. *If* x *varies directly as* y *when* z *is constant, and inversely as* z *when* y *is constant, then* x $\propto \dfrac{\text{y}}{\text{z}}$ *when* y *and* z *are variables.*

Let us suppose that y and z vary in succession, and that in consequence of y becoming y', x becomes x', and that in consequence of z becoming z', x' becomes x''. Then,

(1) $x : x' :: y : y'$, $\therefore$ (3) $\dfrac{x}{x'} = \dfrac{y}{y'}$.

(2) $x' : x'' :: \dfrac{1}{z} : \dfrac{1}{z'}$, $\therefore$ (4) $\dfrac{x'}{x''} = \dfrac{z'}{z}$.

(3) $\times$ (4) $=$ (5) $\dfrac{x}{x''} = \dfrac{z'y}{y'z}$; or $x = \dfrac{x''z'}{y'}\dfrac{y}{z}$, $\therefore$ $x \propto \dfrac{y}{z}$.

321. Examples.

1. If $y = p + q$, and $p \propto x$, and $q \propto x^2$, and when $x = 2$, $y = 16$, and when $x = 3$, $y = 33$, what is the equation between x and y?

SOLUTION.

Since $p \propto x$, $p = mx$, and since $q \propto x^2$, $q = nx^2$.

$$\therefore \quad (1) \quad y = mx + nx^2.$$

Substituting in (1) first 2 for x and 16 for y, then 3 for x and 33 for y, and transposing, we have

$$(2) \quad 2m + 4n = 16.$$

$$(3) \quad 3m + 9n = 33.$$

Solving these equations, we find $m = 2$ and $n = 3$. Substituting the values of m and n in (1), we have

$$(4) \quad y = 2x + 3x^2.$$

2. If $y \propto x$, and when $x = 2$, $y = 4$, what is the equation between x and y? *Ans.* $y = 2x$.

3. If $x \propto y$, and when $x = 12$, $y = 3$, what is the value of x when $y = 5$? *Ans.* $x = 20$.

4. If x varies as the sum of two quantities, one of which varies as y and the other as y^3, and when $y = 3$, $x = 63$, and when $y = 4$, $x = 140$, what is the equation between x and y? *Ans.* $x = 3y + 2y^3$.

5. If x varies as y and z jointly, and $x = 2a$ when $y = 1$ and $z = 2$, what is the equation between x, y, and z? *Ans.* $x = ayz$.

6. If $s \propto t^2$ when f is constant, and as f when t is constant, and when $t = 1$, $f = 2s$, what is the equation between s, f, and t? *Ans.* $s = \frac{1}{2}ft^2$.

7. If x varies as the sum of two quantities, one of which varies directly as y and the other inversely as y, and when $y = 1$, $x = 4$, and when $y = 2$, $x = 5$, what is the equation between x and y? *Ans.* $x = 2y + \frac{2}{y}$.

8. If $x \propto y^m$ and $y \propto z^n$, and when $x = a$, $z = b$, what is the equation between x and z? *Ans.* $x = \frac{a}{b^{mn}} z^{mn}$.

HARMONICAL PROPORTION.

322. Definitions.

1. Three quantities are in harmonical proportion when the first is to the third as the first minus the second is to the second minus the third. Thus, if a, b, c are in harmonical proportion, we shall have

$$a : c :: a - b : b - c.$$

2. Four quantities are in harmonical proportion when the first is to the fourth as the first minus the second is to the third minus the fourth. Thus, if a, b, c, d are in harmonical proportion, we shall have

$$a : d :: a - b : c - d.$$

323. Problems.

1. Find the first term of the harmonical proportion whose second term is b and third c. *Ans.* $\frac{bc}{2c - b}$.

2. Find the second term of the harmonical proportion whose first term is a and third c.

Ans. $\dfrac{2ac}{a+c}$.

3. Find the third term of the harmonical proportion whose first term is a and second b.

Ans. $\dfrac{ab}{2a-b}$.

4. Find the first term of the harmonical proportion whose second term is b, third c, and fourth d.

Ans. $\dfrac{bd}{2d-c}$.

5. Find the second term of the harmonical proportion whose first term is a, third c, and fourth d.

Ans. $\dfrac{2ad-ac}{d}$.

6. Find the third term of the harmonical proportion whose first term is a, second b, and fourth d.

Ans. $\dfrac{2ad-bd}{a}$.

7. Find the fourth term of the harmonical proportion whose first term is a, second b, and third c.

Ans. $\dfrac{ac}{2a-b}$.

ARITHMETICAL PROGRESSION.

324. Definition.

An arithmetical progression is a series in which the difference of the consecutive terms is constant. Thus, 3, 5, 7, . . . , is an arithmetical progression of which the common difference is 2, since $5-3=2$ and $7-5=2\ldots$; and a, $a+d$, $a+2d$, . . . , is an arithmetical progression of which the common difference is d.

325. Elements and Notation.

1. *The first term,* a.
2. *The common difference,* d.
3. *The number of terms,* n.
4. *The last term,* l.
5. *The sum of the terms,* s.

Thus, 5, 8, 11, 14, 17, 20 is an arithmetical progression in which $a=5$, $d=3$, $n=6$, $l=20$, $s=75$. l denotes the last term considered, but the progression may be regarded as extending indefinitely.

326. Classification.

1. Increasing, $d>0$, as 3, 5, 7, ...
2. Decreasing, $d<0$, as 7, 5, 3, ...

327. Problems.

1. Given a, d, and n, required l and s.

1*st*. To find l.

$$\overset{1st.}{a},\quad \overset{2d.}{a \pm d},\quad \overset{3d.}{a \pm 2d},\quad \overset{4th.}{a \pm 3d},\ \ldots\ \overset{nth.}{a \pm (n-1)\,d}.$$

$$\therefore\quad (1)\quad l = a \pm (n-1)\,d.$$

2*d*. To find s.

$$(2)\quad s = a + (a \pm d) + (a \pm 2d) + (a \pm 3d) + \ldots + l.$$

$$(3)\quad s = l + (l \mp d) + (l \mp 2d) + (l \mp 3d) + \ldots + a.$$

$(2) + (3) = (4) \quad 2s = (a + l) + (a + l) + \ldots + (a + l).$

Or $\quad (5) \quad 2s = n(a + l).$

$\therefore \quad (6) \quad s = \frac{1}{2}n(a + l).$

$(1) \text{ in } (6) = (7) \quad s = \frac{1}{2}n[2a \pm (n - 1)d].$

Sch. When the double sign $\pm$ or $\mp$ is used, the upper sign will in general apply when the progression is increasing, and the lower sign when the progression is decreasing.

Thus, if $a = 5$, $d = 3$, $n = 10$, then, by (1) and (7), $l = 5 + 9 \times 3 = 32$, and $s = 5(10 + 9 \times 3) = 185$.

Formulas (1), (6), and (7) are used in solving the following problems:

2. Given a, d, and l, required n and s.

$$Ans. \begin{cases} n = \pm \dfrac{l - a}{d} + 1. \\[2ex] s = \dfrac{(a + l)[d \pm (l - a)]}{2d}. \end{cases}$$

3. Given a, d, and s, required n and l.

$$Ans. \begin{cases} n = \dfrac{d - 2a \pm \sqrt{8ds + (2a - d)^2}}{2d}. \\[2ex] l = \dfrac{-d \pm \sqrt{8ds + (2a - d)^2}}{2}. \end{cases}$$

4. Given a, n, and l, required d and s.

$$Ans. \begin{cases} d = \pm \dfrac{l - a}{n - 1}. \\[2ex] s = \frac{1}{2}n(a + l). \end{cases}$$

5. Given a, n, and s, required l and d.

$$Ans. \begin{cases} l = \dfrac{2s}{n} - a. \\ d = \dfrac{2(s - an)}{n(n-1)}. \end{cases}$$

6. Given a, l, and s, required n and d.

$$Ans. \begin{cases} n = \dfrac{2s}{a + l}. \\ d = \dfrac{(l + a)(l - a)}{2s - (a + l)}. \end{cases}$$

7. Given d, n, and l, required a and s.

$$Ans. \begin{cases} a = l \mp (n - 1)d. \\ s = \frac{1}{2}n[2l \mp (n - 1)d]. \end{cases}$$

8. Given d, n, and s, required a and l.

$$Ans. \begin{cases} a = \dfrac{2s \mp n(n-1)d}{2n}. \\ l = \dfrac{2s \pm n(n-1)d}{2n}. \end{cases}$$

9. Given d, l, and s, required n and a.

$$Ans. \begin{cases} n = \dfrac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}. \\ a = \dfrac{d \mp \sqrt{(2l + d)^2 - 8ds}}{2}. \end{cases}$$

10. Given n, l, and s, required a and d.

$$Ans, \begin{cases} a = \dfrac{2s - ln}{n}. \\ d = \dfrac{2(ln - s)}{n(n-1)}. \end{cases}$$

328. Application.

The answers of the above general problems are *formulas* which may be applied in solving numerical examples.

1. Given $a=5$, $d=3$, and $n=10$, required l and s.
Ans. $l=32$, $s=185$.

2. Given $a=5$, $d=4$, and $l=201$, required n and s.
Ans. $n=50$, $s=5150$.

3. Given $a=4$, $d=3$, and $s=714$, required n and l.
Ans. $n=21$, $l=64$.

4. Given $a=5$, $l=201$, and $n=50$, required d and s.
Ans. $d=4$, $s=5150$.

5. Given $a=7$, $n=10$, and $s=205$, required l and d.
Ans. $l=34$, $d=3$.

6. Given $a=10$, $l=310$, and $s=16160$, required n and d.
Ans. $n=101$, $d=3$.

7. Given $d=5$, $n=100$, and $l=498$, required a and s.
Ans. $a=3$, $s=25050$.

8. Given $d=10$, $n=1000$, and $s=10003000$, required a and l.
Ans. $a=5008$, $l=14998$.

9. Given $d=5$, $l=104$, and $s=1134$, required n and a.
Ans. $n=21$, $a=4$.

10. Given $n=12$, $l=40$, and $s=282$, required a and d.
Ans. $a=7$, $d=3$.

329. Problem.

To insert any number of arithmetical means between any two quantities.

The means together with the extremes form an arithmetical progression; and if we knew the common difference, the progression could be written out. The number of terms equals the number of means plus 2. Let $m =$ the number of means, then $n = m + 2$. But according to problem 4,

$$d = \frac{l - a}{n - 1}.$$

Substituting the value of n, we have

$$d = \frac{l - a}{m + 1}.$$

1. Insert 8 means between 3 and 30.

$$d = \frac{30 - 3}{9} = 3; \quad \therefore \quad 3, 6, 9, 12, 15, 18, 21, 24, 27, 30.$$

2. Insert 12 means between 12 and 77.

3. Insert 10 means between 82 and 5.

4. Insert 9 means between the consecutive terms of the progression 2, 5, 8, and write out the whole progression.

330. Miscellaneous Problems.

In certain problems in arithmetical progression, if the number of terms is odd, it is convenient to represent the middle term by x, and the common difference by y; but

if the number of terms is even, to represent the two middle terms by $x - y$ and $x + y$, and the common difference by $2y$, as follows:

Three terms, $x - y,\ x,\ x + y$.

Four terms, $x - 3y,\ x - y,\ x + y,\ x + 3y$.

Five terms, $x - 2y,\ x - y,\ x,\ x + y,\ x + 2y$.

1. The sum of three numbers in arithmetical progression is 24, and the sum of their squares 200; required the numbers. *Ans.* 6, 8, 10.

2. The sum of four numbers in arithmetical progression is 16, and the sum of their squares is 84; required the numbers. *Ans.* 1, 3, 5, 7.

3. The sum of five numbers in arithmetical progression is 30, and the sum of their squares is 220; required the numbers. *Ans.* 2, 4, 6, 8, 10.

4. The sum of six numbers in arithmetical progression is 36, and the sum of their squares is 286; required the numbers. *Ans.* 1, 3, 5, 7, 9, 11.

5. The product of four numbers in arithmetical progression is 384, the sum of their squares is 120; required the numbers. *Ans.* 2, 4, 6, 8.

6. In the series of odd numbers, find the n^{th} term and the sum of n terms. *Ans.* $l = 2n - 1$, $s = n^2$.

7. How many terms of the series 16, 14, 12, etc., must be taken in order that the sum be 60? *Ans.* 5 or 12.

GEOMETRICAL PROGRESSION.

331. Definition.

A geometrical progression is a series in which the ratio of the consecutive terms is constant. Thus, 2, 6, 18, ..., is a geometrical progression, of which the ratio is 3; a, ar, ar^2, ar^3, ..., a geometrical progression, of which the ratio is r.

332. Elements and Notation.

1. *The first term,* a.
2. *The ratio,* r.
3. *The number of terms,* n.
4. *The last term,* l.
5. *The sum of the terms,* s.

Thus, 3, 6, 12, 24, 48 is a geometrical progression in which $a = 3$, $r = 2$, $n = 5$, $s = 93$.

333. Classification.

1. Increasing, $r > 1$; as 1, 3, 9, ...
2. Decreasing, $r < 1$; as 21, 7, $2\frac{1}{3}$, ...

334. Problems.

1. Given a, r, and n, required l and s.

1*st*. To find l.

$$\overset{1st.}{a},\quad \overset{2d.}{ar},\quad \overset{3d.}{ar^2},\quad \overset{4th.}{ar^3},\ \ldots\quad \overset{nth.}{ar^{n-1}}.$$

$$\therefore\quad l = ar^{n-1}.$$

2*d*. To find s.

$$(1)\quad s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}.$$

$$(2)\quad rs = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n.$$

$$(2) - (1) = (3)\quad (r-1)\, s = a\,(r^n - 1).$$

$$\therefore\quad (4)\quad s = \frac{a\,(r^n - 1)}{r - 1}.$$

Thus, if $a = 4$, $r = 3$, $n = 10$, then we have $l = 4 \times 3^9 = 78732$, $s = \dfrac{4\,(3^{10} - 1)}{2} = 118096$.

2. Given a, r, and l, required s and n.

Ans. $s = \dfrac{lr - a}{r - 1}$.

To find n: since $l = ar^{n-1}$, $r^{n-1} = \dfrac{l}{a}$; then raise r to a power which equals $\dfrac{l}{a}$, and the exponent of this power will be $n - 1$. Denoting the exponent by e, we have $n - 1 = e$, $\therefore\ n = e + 1$.

3. Given a, r, and s, required l and n.

Ans. $l = \dfrac{(r-1)\, s + a}{r}$.

After finding l, find n, as in problem 2.

4. Given a, n, and l, required r and s.

$$\textit{Ans.}\ \begin{cases} r = \sqrt[n-1]{\dfrac{l}{a}}. \\[2ex] s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}. \end{cases}$$

5. Given a, n, and s, required the equations involving r and l.

$$\textit{Ans.}\ \begin{cases} r^n - \dfrac{s}{a}\, r + \dfrac{s - a}{a} = 0. \\[2ex] l\,(s - l)^{n-1} - a\,(s - a)^{n-1} = 0. \end{cases}$$

6. Given a, l, and s, required r and n.

$$\textit{Ans.}\ r = \frac{s-a}{s-l}.$$

After finding r, find n, as in problem 2.

7. Given r, n, and l, required a and s.

$$\textit{Ans.}\ \begin{cases} a = \dfrac{l}{r^{n-1}}. \\[2ex] s = \dfrac{l\,(r^n - 1)}{(r-1)\,r^{n-1}}. \end{cases}$$

8. Given r, n, and s, required a and l.

$$\textit{Ans.}\ \begin{cases} a = \dfrac{(r-1)\,s}{r^n - 1}. \\[2ex] s = \dfrac{s\,(r-1)\,r^{n-1}}{r^n - 1}. \end{cases}$$

9. Given r, l, and s, required a and n.

$$\textit{Ans.}\ a = lr - (r-1)\,s.$$

After finding a, find n, as in problem 2.

10. Given n, l, and s, required the equations involving a and r.

$$\textit{Ans.}\ \begin{cases} a\,(s-a)^{n-1} - l\,(s-l)^{n-1} = 0. \\[2ex] r^n - \dfrac{s}{s-l}\,r^{n-1} + \dfrac{l}{s-l} = 0. \end{cases}$$

335. Application.

1. Given $a = 5$, $r = 3$, and $n = 10$, required l and s.

Ans. $l = 98415$, $s = 147620$.

2. Given $a = 10$, $r = 10$, $l = 100000000000$, required n and s. *Ans.* $n = 11$, $s = 111111111110$.

3. Given $a=1$, $r=5$, and $s=97656$, required n and l.
Ans. $l=78125$, $n=8$.

4. Given $a=3$, $n=5$, and $l=768$, required r and s.
Ans. $r=4$, $s=1023$.

5. Given $a=3$, $n=5$, and $s=1023$, required r and l.
Ans. $r=4$, $l=768$.

6. Given $a=4$, $l=12500$, and $s=15624$, required r and n.
Ans. $r=5$, $n=6$.

7. Given $r=2$, $n=10$, and $l=2560$, required a and s.
Ans. $a=5$, $s=5115$.

8. Given $r=10$, $n=9$, and $s=1111111110$, required a and l.
Ans. $a=10$, $l=1000000000$.

9. Given $r=10$, $l=500000$, and $s=555555$, required a and n.
Ans. $a=5$, $n=6$.

10. Given $n=6$, $l=12500$, and $s=15624$, required r and a.
Ans. $r=5$, $a=4$.

336. Problem.

To insert any number of geometrical means between any two quantities.

The means together with the extremes form a geometrical progression; and if we knew the ratio, the progression could be written out. The number of terms equals the number of means plus 2. Let $m=$ the number of means, then $n=m+2$. But according to problem 4,

$$r=\sqrt[n-1]{\frac{l}{a}}.$$

Substituting the value of n, we have

$$r = \sqrt[m+1]{\frac{l}{a}}.$$

1. Insert 5 means between 5 and 3645.

$$r = \sqrt[6]{\frac{3645}{5}} = 3; \quad \therefore \quad 5,\ 15,\ 45,\ 135,\ 405,\ 1215,\ 3645.$$

2. Insert 2 means between 8 and 512.

3. Insert 4 means between 10 and 1000000.

4. Insert 1 mean between the consecutive terms of the progression 6, 150, 3750, and write out the whole progression.

337. Problem.

To find the sum of a decreasing geometrical progression having an infinite number of terms.

If $r < 1$ and $n = \infty$, $r^n = 0$. Substituting this value of r^n in the formula

$$s = \frac{a\,(r^n - 1)}{r - 1},$$

we have

$$s = \frac{-a}{r-1}.$$

Changing the signs of both terms of the fraction, we have

$$s = \frac{a}{1-r}.$$

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, *ad infinitum* $=1$.

2. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$, *ad infinitum* $=\frac{1}{2}$.

3. $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots$, *ad infinitum* $=\frac{1}{3}$.

4. $\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\ldots$, *ad infinitum* $=\frac{1}{n-1}$.

338. Miscellaneous Problems.

1. Find six numbers in geometrical progression whose sum is 252, and the sum of whose extremes is 132.

SOLUTION.

$$(1) \quad a+ar+ar^2+ar^3+ar^4+ar^5=252.$$

$$(2) \quad a+ar^5=132.$$

$$(1)-(2)=(3) \quad ar+ar^2+ar^3+ar^4=120.$$

$$(1)=(4) \quad \frac{a(r^6-1)}{r-1}=252.$$

$$\therefore \quad (5) \quad a=\frac{252(r-1)}{r^6-1}.$$

$$(3)=(6) \quad \frac{ar(r^4-1)}{r-1}=120.$$

$$\therefore \quad (7) \quad a=\frac{120(r-1)}{r(r^4-1)}.$$

$$(5) \text{ and } (7) \text{ give } (8) \quad \frac{120(r-1)}{r(r^4-1)}=\frac{252(r-1)}{r^6-1}.$$

(8) gives (9) $\dfrac{10}{r(r^2+1)} = \dfrac{21}{r^4+r^2+1}$.

$\therefore$ (10) $10(r^2+1)^2 - 10r^2 = 21r(r^2+1)$.

Or, (11) $10(r^2+1)^2 - 21r(r^2+1) = 10r^2$.

$\therefore$ (12) $r^2+1 = \dfrac{21r \pm \sqrt{441r^2+400r^2}}{20} = \frac{5}{2}r$.

Or, (13) $r^2 - \frac{5}{2}r = -1$.

$\therefore$ (14) $r = 2$ or $\frac{1}{2}$.

(14) in (2) gives $a = 4$ or 128. Hence, the progressions:

4, 8, 16, 32, 64, 128, or 128, 64, 32, 16, 8, 4.

In certain problems in geometrical progression it is convenient to represent the ratio by $\dfrac{y}{x}$, and the middle term by xy, if the number of terms is odd; and the two middle terms by x and y, if the number of terms is even, as follows:

Three terms, thus: x^2, xy, y^2,

Four terms, thus: $\dfrac{x^2}{y}$, x, y, $\dfrac{y^2}{x}$.

Five terms, thus: $\dfrac{x^3}{y}$, x^2, xy, y^2, $\dfrac{y^3}{x}$.

2. Find 3 numbers in geometrical progression whose continued product is 216, and the sum of their cubes 1971.

Ans. 3, 6, 12.

3. Find 4 numbers in geometrical progression the sum of whose extremes is 45, and the sum of whose means is 30.

Ans. 5, 10, 20, 40.

4. Find 4 numbers in geometrical progression the sum of whose first and third terms is 40, and the sum of the second and fourth 120.

Ans. 4, 12, 36, 108.

5. Find 4 numbers in arithmetical progression which being increased by 1, 3, 7, 14, respectively, the sums will be in geometrical progression.

Ans. 7, 9, 11, 13.

6. Find 3 numbers in geometrical progression whose sum is 13, and the sum of whose reciprocals is $\frac{13}{9}$.

Ans. 1, 3, 9.

7. Find 3 numbers in geometrical progression whose sum is 14, and the sum of whose squares is 84.

Ans. 2, 4, 8.

8. Find 4 numbers in geometrical progression whose sum is 15, and the sum of whose squares is 85.

Ans. 1, 2, 4, 8.

9. Find 6 numbers in geometrical progression whose sum is 126, and the sum of the means 60.

Ans. 2, 4, 8, 16, 32, 64.

10. Find 6 numbers in geometrical progression the sum of whose extremes is 488, and of the means 240.

Ans. 2, 6, 18, 54, 162, 486.

HARMONICAL PROGRESSION.

339. Definition.

An harmonical progression is a series in which the first of any three consecutive terms is to the third as the first minus the second is to the second minus the third.

340. Propositions.

1. *The reciprocals of the terms of an harmonical progression are in arithmetical progression.*

Let a, b, c, d . . . be in harmonical progression; then,

(1) $a : c :: a - b : b - c$. $\therefore$ (3) $ab - ac = ac - bc$.

(2) $b : d :: b - c : c - d$. $\therefore$ (4) $bc - bd = bd - cd$.

We find that (3) $\div\, abc =$ (5) $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$.

We find that (4) $\div\, bcd =$ (6) $\frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}$.

$\therefore\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \ldots$ are in arithmetical progression.

2. *The reciprocals of the terms of an arithmetical progression are in harmonical progression.*

Let $a, b, c, d \ldots$ be in arithmetical progression; then,

$$(1) \quad c - b = b - a.$$

$$(1) \div abc = (3) \quad \frac{1}{a}\left(\frac{1}{b} - \frac{1}{c}\right) = \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right).$$

$$(2) \quad d - c = c - b.$$

$$(2) \div bcd = (4) \quad \frac{1}{b}\left(\frac{1}{c} - \frac{1}{d}\right) = \frac{1}{d}\left(\frac{1}{b} - \frac{1}{c}\right).$$

From (3) we have $\quad (5) \quad \frac{1}{a} : \frac{1}{c} :: \frac{1}{a} - \frac{1}{b} : \frac{1}{b} - \frac{1}{c}.$

From (4) we have $\quad (6) \quad \frac{1}{b} : \frac{1}{d} :: \frac{1}{b} - \frac{1}{c} : \frac{1}{c} - \frac{1}{d}.$

$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \ldots$ are in harmonical progression.

3. *The geometrical mean of two quantities is a mean proportional between their arithmetical and harmonical means.*

Let p and q be two quantities, g their geometrical mean, a their arithmetical mean, and h their harmonical mean; then,

$$(1) \quad p : g :: g : q. \qquad \therefore \quad (4) \quad g^2 = pq.$$

$$(2) \quad a - p = q - a. \qquad \therefore \quad (5) \quad a = \tfrac{1}{2}(p + q).$$

$$(3) \quad p : q :: p - h : h - q. \quad \therefore \quad (6) \quad h = \frac{2pq}{p + q}.$$

$$(5) \times (6) = (7) \quad ah = pq.$$

Comparing (4) and (7), we have $\quad (8) \quad g^2 = ah.$

$$\therefore \quad (9) \quad a : g :: g : h.$$

341. Problems.

1. Insert m harmonical means between a and l.

SOLUTION.

The problem is solved by inserting m arithmetical means between $\frac{1}{a}$ and $\frac{1}{l}$, and taking the reciprocals.

2. Insert 1 harmonic mean between 20 and 60.

Ans. 20, 30, 60.

3. Insert 2 harmonic means between 12 and 30.

Ans. 12, 15, 20, 30.

4. Insert 4 harmonic means between 10 and 60.

Ans. 10, 12, 15, 20, 30, 60.

5. Continue the harmonic series 10, 12, 15.

Ans. 10, 12, 15, 20, 30, 60, ∞,
-60, -30, -20, -15, -12, -10.

6. Find the n^{th} term of the harmonical progression $a, b, \ldots$

Ans. $\dfrac{ab}{(n-1)a-(n-2)b}$.

7. If a, b, c are in arithmetical progression, and b, c, d in harmonical progression, prove that $a : b :: c : d$.

8. Prove that the arithmetical mean of two unequal quantities is greater than the harmonic mean.

9. What quantity added to each of the quantities a, b, c will give results in harmonical proportion?

Ans. $\dfrac{ab-2ac+bc}{a-2b+c}$.

10. If a is the arithmetical mean of x and y, and h their harmonical mean, find x and y.

$$Ans.\quad \begin{cases} x = a \pm \sqrt{a^2 - ah}. \\ y = a \mp \sqrt{a^2 - ah}. \end{cases}$$

11. If g is the geometrical mean of x and y, and h their harmonical mean, find x and y.

$$Ans.\quad \begin{cases} x = \frac{g}{h}(g \pm \sqrt{g^2 - h^2}). \\ y = \frac{g}{h}(g \mp \sqrt{g^2 - h^2}). \end{cases}$$

PERMUTATIONS.

342. Definition.

Permutations are the results obtained by placing a certain number of things in every possible order in sets of 1, 2, 3,... r,... n.

343. Problem.

To find the number of permutations of n *letters taken* 1, 2, 3, .. r, ... n *in a set.*

Let P_1, P_2, P_3, ... P_r, ... P_n, respectively, denote the number of permutations of n letters taken 1, 2, 3,... r,... n in a set. n letters taken 1 in a set give n permutations.

$$\therefore \quad (1) \quad P_1 = n.$$

For each set of 1 letter there are $n - 1$ remaining letters which may be annexed singly to the corresponding set of 1, thus giving for each set of 1, $n - 1$ sets of 2; hence, for the n sets of 1, there are $n(n-1)$ sets of 2.

$$\therefore \quad (2) \quad P_2 = n(n-1).$$

For each set of 2 letters there are $n - 2$ remaining letters which may be annexed singly to the corresponding set of 2, thus giving for each set of 2, $n - 2$ sets of 3; hence, for the $n(n-1)$ sets of 2, there are $n(n-1)(n-2)$ sets of 3.

$$\therefore \quad (3) \quad P_3 = n(n-1)(n-2).$$

Thus far, the following law holds true:

The factors are n, n − 1, n − 2, *the negative number in the last factor being less by unity than the number of letters in a set.*

To prove the law general, let us assume it true for $r - 1$ in a set. The last factor will be $n - (r-2)$ or $n - r + 2$.

$$\therefore (r-1) \quad P_{r-1} = n(n-1)(n-2)\dots(n-r+2).$$

For each set of $r - 1$ letters there are $n - (r-1)$ or $n - r + 1$ remaining letters which may be annexed singly to the corresponding set of $r - 1$ letters, thus giving, for each set of $r - 1$, $n - r + 1$ sets of r; hence, for the $n(n-1)(n-2)\dots(n-r+2)$ sets of $r - 1$, there are $n(n-1)(n-2)\dots(n-r+2)(n-r+1)$ sets of r.

$$\therefore (r) \quad P_r = n(n-1)(n-2)\dots(n-r+2)(n-r+1).$$

Hence, if the law holds for n letters taken $r - 1$ in a set, it will hold for n letters taken r in a set. But we have

found that it holds for 3 in a set; hence, it will hold for 4 in a set; and if for 4, then for 5, and so on up to r.

If $r = n$, all the letters will be taken in each set; and by changing the order of the factors, (r) will become

$$(n) \quad P_n = 1.2.3 \ldots n.$$

Hence, the number of permutations of n *letters taken* n *in a set is equal to the product of the numbers from* 1 *up to* n *inclusive.*

For the sake of brevity, we shall denote $1.2.3 \ldots n$ by $\underline{|n}$, which is read *factorial* n. Then (n) becomes

$$P_n = \underline{|n}.$$

COMBINATIONS.

344. Definition.

Combinations are the results obtained by placing a certain number of things in sets of $1, 2, 3, \ldots r, \ldots n$, any two sets differing from each other by at least one of the things.

345. Problem.

To find the number of combinations of n *letters taken* 1, 2, 3, ... r, ... n *in a set.*

Let $C_1, C_2, C_3, \ldots C_r, \ldots C_n$, respectively, denote the number of combinations of n letters taken $1, 2, 3, \ldots r, \ldots n$ in a set.

To ascertain the law of relation between combinations and permutations, let us take 3 letters, a, b, c, two in a set. Then,

$$\text{Combinations,}\ \begin{cases} ab. \\ ac. \\ bc. \end{cases} \qquad \text{Permutations,}\ \begin{cases} ab. \\ ba. \\ ac. \\ ca. \\ bc. \\ cb. \end{cases}$$

The combinations are converted into permutations by permuting the letters of each combination.

Now, suppose the C_r combinations formed, then each of these combinations will give $\underline{|r}$ permutations; hence, the C_r combinations will give $\underline{|r} \times C_r$ permutations; hence,

$$\underline{|r} \times C_r = P_r. \quad \therefore \quad C_r = \frac{P_r}{\underline{|r}}.$$

Hence,

$$(1) \quad C_1 = \frac{P_1}{\underline{|1}} = \frac{n}{1}.$$

$$(2) \quad C_2 = \frac{P_2}{\underline{|2}} = \frac{n(n-1)}{1 \cdot 2}.$$

$$(3) \quad C_3 = \frac{P_3}{\underline{|3}} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}.$$

. . . .

$$(r) \quad C_r = \frac{P_r}{\underline{|r}} = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \dots r}.$$

. . . .

$$(n) \quad C_n = \frac{P_n}{\underline{|n}} = \frac{n(n-1)(n-2)\dots 2 \cdot 1}{1 \cdot 2 \cdot 3 \dots (n-1)\,n}.$$

346. Proposition.

The number of combinations of n *things taken* r *in a set is equal to the number of combinations of* n *things taken* n — r *in a set.*

This is evident from the fact that for each combination of n things taken r in a set there is a complementary combination taken $n - r$ in a set.

347. Problems.

1. *To find the number of permutations of* n *letters taken all in a set, if* p *of these letters are of one kind,* q *of another kind,* s *of another, and so on.*

The p letters which are alike would, if different, give for each position of the other letters $\underline{|p}$ permutations; but since these letters are alike, $\underline{|p}$ permutations reduce to one; hence, there will be $\frac{1}{\underline{|p}}$ as many permutations when the p letters are alike as when they are different.

In like manner, there will be $\frac{1}{\underline{|q}}$ as many permutations when the q letters are alike as when they are different.

Also, there will be $\frac{1}{\underline{|s}}$ as many permutations when the s letters are alike as when they are different.

But if the p letters were different, also the q letters and the s letters, . . . the number of permutations of the n letters would be $\underline{|n}$. Hence, when the p letters are alike, also the

q letters and the s letters, . . . the number of permutations of n letters taken all in a set is

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor s \ldots}.$$

2. *To find the number of sets of* n *different letters when each may occur* 1, 2, 3, . . . r *times.*

The number of sets of 1 in a set is n.

Placing each letter before each set, the number of sets of 2 in a set is n^2.

Placing each letter before each set, the number of sets of 3 in a set is n^3.

In general, the number of sets of r in a set is n^r.

348. Examples.

1. How many permutations can be made of 6 letters taken 1, 2, 3, 4, 5, 6 in a set?

Ans. 6, 30, 120, 360, 720, 720.

2. What is the total number of permutations that can be made of 10 letters? *Ans.* 9864100.

3. How many combinations can be made of 8 letters taken in sets of 1, 2, 3, 4, 5, 6, 7, 8?

Ans. 8, 28, 56, 70, 56, 28, 8, 1.

4. What is the total number of combinations that can be made of 12 letters? *Ans.* 4095.

5. How many permutations, taken all in a set, can be made of the letters in the word Algebra? *Ans.* 2520.

6. Prove that the product of any r consecutive numbers is divisible by $\underline{|r}$.

7. Prove that the number of combinations of n things, taken r in a set, is equal to *factorial* n divided by *factorial* r into *factorial* $n-r$.

INDETERMINATE CO-EFFICIENTS

349. Theorem I.

If $Ax^a + Bx^b + Cx^c + \ldots = A'x^{a'} + B'x^{b'} + C'x^{c'} + \ldots,$ *for all possible values of* x, *the co-efficients and exponents being finite quantities, and independent of* x, *and the terms arranged according to the ascending powers of* x, *then* $a = a'$, $b = b'$, $c = c'$, ... *and* $A = A'$, $B = B'$, $C = C'$...

Since the co-efficients and exponents are independent of x, if we find their values for one value of x, we shall have their values for all values of x.

(1) $Ax^a + Bx^b + Cx^c + \ldots = A'x^{a'} + B'x^{b'} + C'x^{c'} + \ldots$

1. Suppose $a < a'$.

Dividing (1) by x^a, we shall have

(2) $A + Bx^{b-a} + Cx^{c-a} + \ldots = A'x^{a'-a} + B'x^{b'-a} + C'x^{c'-a} + \ldots$

The exponents of x are all positive; and making $x=0$, we find $A=0$, which is contrary to the hypothesis that the co-efficients are finite quantities. Therefore, it is not true that $a < a'$.

2. Suppose $a > a'$.

Dividing (1) by $x^{a'}$, we shall have

$$(3)\quad Ax^{a-a'} + Bx^{b-a'} + Cx^{c-a'} + \ldots = A' + B'x^{b'-a'} + C'x^{c'-a'} + \ldots$$

The exponents of x are all positive; and making $x = 0$, we find $0 = A'$, which is contrary to the hypothesis. Therefore, it is not true that $a > a'$.

Since neither $a < a'$ nor $a > a'$, it follows that

$$(4)\quad a = a'. \quad \therefore \quad (5)\quad x^a = x^{a'}.$$

Dividing (1) by (5), member by member, we have

$$(6)\quad A + Bx^{b-a} + Cx^{c-a} + \ldots = A' + B'x^{b'-a'} + C'x^{c'-a'} + \ldots$$

Making $x = 0$, we shall have

$$(7)\quad A = A'. \quad (5) \times (7) = (8)\quad Ax^a = A'x^{a'}.$$

Subtracting (8) from (1), member from member, we have

$$(9)\quad Bx^b + Cx^c + \ldots = B'x^{b'} + C'x^{c'} + \ldots$$

Since (9) is the same in form as (1), by a similar process it can be proved that $b = b'$ and $B = B'$; and, in like manner, that $c = c'$ and $C = C'$. and so on.

350. Theorem II.

If $Ax^a + Bx^b + Cx^c + Dx^d + \ldots = 0$, for all possible values of x, *the co-efficients and exponents involved in the equation being independent of* x, *and the terms arranged according*

to the ascending powers of x, *then* $A = 0$, $B = 0$, $C = 0$, $D = 0$, *and so on.*

$$(1)\quad Ax^a + Bx^b + Cx^c + Dx^d + \ldots = 0.$$

Dividing (1) by x^a, we shall have

$$(2)\quad A + Bx^{b-a} + Cx^{c-a} + Dx^{d-a} + \ldots = 0.$$

Making $x = 0$, we shall have

$$(3)\quad A = 0. \quad \therefore \quad (4)\quad Ax^a = 0.$$

Subtracting (4) from (1), member from member, we have

$$(5)\quad Bx^b + Cx^c + Dx^d + \ldots = 0.$$

Since (5) is the same in form as (1), by a similar process it can be proved that $B = 0$; and, in like manner, that $C = 0$, $D = 0$, and so on.

351. Expansion of Fractions into Series.

1. Expand $\dfrac{4}{2+3x}$ into a series.

1ST SOLUTION.—BY DIVISION.

$$\frac{4}{2+3x} = 2 - 3x + \frac{9}{2}x^2 - \frac{27}{4}x^3 + \ldots$$

2D SOLUTION.—BY INDETERMINATE CO-EFFICIENTS.

It is evident by division that the first term of the development will be independent of x; $\therefore$ assume

$$(1)\quad \frac{4}{2+3x} = A + Bx + Cx^2 + Dx^3 + \ldots$$

Clearing of fractions and transposing, we have

$$(2)\quad 0 = \begin{array}{c|c|c|c|l} 2A & +2B & x + 2C & x^2 + 2D & x^3 + \dots \\ -4 & +3A & +3B & +3C & \end{array}$$

Hence, by Theorem 2, we have

$$2A - 4 = 0. \quad \therefore \quad A = 2.$$
$$2B + 3A = 0. \quad \therefore \quad B = -\tfrac{3}{2}A = -3.$$
$$2C + 3B = 0. \quad \therefore \quad C = -\tfrac{3}{2}B = \tfrac{9}{2}.$$
$$2D + 3C = 0. \quad \therefore \quad D = -\tfrac{3}{2}C = -\tfrac{27}{4}.$$
$$\dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots\dots$$

Substituting these values in (1), we have

$$(3)\quad \frac{4}{2+3x} = 2 - 3x + \frac{9}{2}x^2 - \frac{27}{4}x^3 + \dots$$

Any term can be obtained by multiplying the preceding by $-\frac{3}{2}x$, which is the scale of the series.

2. Expand $\dfrac{5}{3x^2 - 4x^3}$ into a series.

We find by division that the first term of the expansion is $\dfrac{5}{3x^2} = \dfrac{5}{3}x^{-2}$. Let us therefore assume

$$(1)\quad \frac{5}{3x^2 - 4x^3} = Ax^{-2} + Bx^{-1} + Cx^0 + Dx + Ex^2 + \dots$$

Proceeding as before, we shall find

$$(2)\quad \frac{5}{3x^2 - 4x^3} = \frac{5}{3x^2} + \frac{20}{9x} + \frac{80}{27} + \frac{320}{81}x + \frac{1280}{243}x^2 + \dots$$

3. Expand $\dfrac{1}{1+x}$ into a series.

Ans. $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

4. Expand $\dfrac{1}{1-x}$ into a series.

Ans. $1 + x + x^2 + x^3 + x^4 + x^5 + \ldots$

5. Expand $\dfrac{2}{3+4x}$ into a series.

Ans. $\dfrac{2}{3} - \dfrac{8}{9}x + \dfrac{32}{27}x^2 - \dfrac{128}{81}x^3 + \ldots$

6. Expand $\dfrac{a}{b+cx}$ into a series.

Ans. $\dfrac{a}{b} - \dfrac{ac}{b^2}x + \dfrac{ac^2}{b^3}x^2 - \dfrac{ac^3}{b^4}x^3 + \ldots$

7. Expand $\dfrac{1+x}{1+2x+3x^2}$ into a series.

Ans. $1 - x - x^2 + 5x^3 - 7x^4 - \ldots$

352. Decomposition of Rational Fractions.

1. Let us take the fraction $\dfrac{ax+c}{x^2-b^2}$.

Since $x^2 - b^2 = (x+b)(x-b)$, we may assume

$$(1) \quad \frac{ax+c}{x^2-b^2} = \frac{A}{x+b} + \frac{B}{x-b}.$$

Clearing of fractions and reducing, we have

$$(2) \quad ax + c = (A+B)x - bA + bB.$$

Hence, by the principle of indeterminate co-efficients,

$$\left.\begin{aligned} A + B &= a. \\ -bA + bB &= c. \end{aligned}\right\} \quad \therefore \quad \left\{\begin{aligned} A &= \frac{ab - c}{2b}. \\ B &= \frac{ab + c}{2b}. \end{aligned}\right.$$

Substituting the values of A and B in (1), we have

$$\frac{ax + c}{x^2 - b^2} = \frac{ab - c}{2b\,(x + b)} + \frac{ab + c}{2b\,(x - b)}.$$

2. Decompose $\dfrac{5x + 4}{x^2 - 7x + 12}$. *Ans.* $\dfrac{24}{x - 4} - \dfrac{19}{x - 3}$.

3. Decompose $\dfrac{2x - 3}{x^2 - 3x + 2}$. *Ans.* $\dfrac{1}{x - 1} + \dfrac{1}{x - 2}$.

4. Decompose $\dfrac{5x + 32}{x^2 + 13x + 42}$. *Ans.* $\dfrac{2}{x + 6} + \dfrac{3}{x + 7}$.

THE BINOMIAL THEOREM.

353. Lemma.

$$\left(\frac{x^n - y^n}{x - y}\right)_{y=x} = nx^{n-1}, \text{ for all values of } n.$$

The above is read $\dfrac{x^n - y^n}{x - y}$, when $y = x$, $= nx^{n-1}$.

1. When n is a positive integer.

By division, we shall have

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}\,y + x^{n-3}\,y^2 + \ldots + y^{n-1}.$$

There are n terms in the second member, since there are $n-1$ terms involving y and one term involving x only; but when $y=x$, each term $=x^{n-1}$.

$$\therefore \quad \left(\frac{x^n-y^n}{x-y}\right)_{y=x} = nx^{n-1}, \text{ when } n \text{ is a positive integer.}$$

2. When $n=\frac{p}{q}$, a positive fraction.

Let $x=r^q$, $\therefore\ x^{\frac{p}{q}}=r^p$, and $r^{p-q}=x^{\frac{p}{q}-1}$.

Let $y=s^q$, $\therefore\ y^{\frac{p}{q}}=s^p$, and if $s=r$, $y=x$.

$$\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}=\frac{r^p-s^p}{r^q-s^q}=\left\{\frac{\dfrac{r^p-s^p}{r-s}}{\dfrac{r^q-s^q}{r-s}}\right\}_{s=r}=\frac{pr^{p-1}}{qr^{q-1}}$$

$$=\frac{p}{q}r^{p-q}=\frac{p}{q}x^{\frac{p}{q}-1}; \quad \therefore \left\{\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}\right\}_{y=x}=\frac{p}{q}x^{\frac{p}{q}-1}.$$

3. When n is a negative integer.

$$\frac{x^{-n}-y^{-n}}{x-y}=-x^{-n}y^{-n}\left(\frac{x^n-y^n}{x-y}\right)_{y=x}=-x^{-2n}\times nx^{n-1}$$

$$=-nx^{-n-1}; \quad \therefore \left(\frac{x^{-n}-y^{-n}}{x-y}\right)_{y=x}=-nx^{-n-1}.$$

4. When $n=-\frac{p}{q}$, a negative fraction.

$$\frac{x^{-\frac{p}{q}}-y^{-\frac{p}{q}}}{x-y}=-x^{-\frac{p}{q}}y^{-\frac{p}{q}}\left(\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}\right)_{y=x}=-x^{-\frac{2p}{q}}\times\frac{p}{q}x^{\frac{p}{q}-1}$$

$$=-\frac{p}{q}x^{-\frac{p}{q}-1}; \quad \therefore \left(\frac{x^{-\frac{p}{q}}-y^{-\frac{p}{q}}}{x-y}\right)_{y=x}=-\frac{p}{q}x^{-\frac{p}{q}-1}.$$

354. Application.

Let us now find the development of

$$(a + x)^n,$$

when n is integral or fractional, positive or negative.

Let us assume

$$(1) \quad (a + x)^n = A + Bx + Cx^2 + Dx^3 + \ldots$$

Making $x = 0$, we find $a^n = A$; hence,

$$(2) \quad (a + x)^n = a^n + Bx + Cx^2 + Dx^3 + \ldots$$

Substituting y for x, we have

$$(3) \quad (a + y)^n = a^n + By + Cy^2 + Dy^3 + \ldots$$

$$(4) \quad (a + x) - (a + y) = x - y.$$

Subtracting (3) from (2), and dividing by (4),

$$(5) \quad \frac{(a + x)^n - (a + y)^n}{(a + x) - (a + y)} = B\left(\frac{x - y}{x - y}\right) + C\left(\frac{x^2 - y^2}{x - y}\right) + D\left(\frac{x^3 - y^3}{x - y}\right) + \ldots$$

But, for all values of n, when $y = x$, (5) becomes

$$(6) \quad n(a + x)^{n-1} = B + 2Cx + 3Dx^2 + \ldots$$

Multiplying both members of (6) by $(a + x)$, we have

$$(7) \quad n(a + x)^n = aB + \left.\begin{matrix} 2aC \\ + B \end{matrix}\right| x + \left.\begin{matrix} 3aD \\ + 2C \end{matrix}\right| x^2 + \ldots$$

Multiplying both members of (2) by n, we have

$$(8) \quad n\,(a+x)^n = na^n + nBx + nCx^2 + \dots$$

Equating the second members of (7) and (8),

$$(9) \quad \left.\begin{array}{r} aB + 2aC \\ + B \end{array}\right| x + \left.\begin{array}{r} 3aD \\ + 2C \end{array}\right| x^2 + \dots = na^n + nBx + nCx^2 + \dots$$

Hence, by the principle of Indeterminate Co-efficients,

$$aB = na^n, \therefore\ B = na^{n-1}.$$

$$2aC + \ \ B = nB, \therefore\ C = \frac{(n-1)B}{2a} = \frac{n\,(n-1)}{1\ .\ 2}\,a^{n-2}.$$

$$3aD + 2C = nC, \therefore\ D = \frac{(n-2)C}{3a} = \frac{n\,(n-1)\,(n-2)}{1\ .\ 2\ .\ 3}\,a^{n-3}.$$

. . . .

Substituting the values of B, C, D, . . . in (2), we have, for any exponent, the Binomial Formula,

$$(a+x)^n = a^n + na^{n-1}\,x + \frac{n\,(n-1)}{1\ .\ 2}\,a^{n-2}\,x^2 + \frac{n\,(n-1)\,(n-2)}{1\ .\ 2\ .\ 3}\,a^{n-3}\,x^3 + \dots$$

355. Examples.

1. $(a+x)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}\,a^{\frac{1}{2}-1}\,x + \frac{\frac{1}{2}\,(\frac{1}{2}-1)}{1\ .\ 2}\,a^{\frac{1}{2}-2}\,x^2.$

$$+ \frac{\frac{1}{2}\,(\frac{1}{2}-1)\,(\frac{1}{2}-2)}{1\ .\ 2\ .\ 3}\,a^{\frac{1}{2}-3}\,x^3 + \dots$$

$$= a^{\frac{1}{2}} + \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} + \frac{x^3}{16a^{\frac{5}{2}}} - \frac{x^4}{128a^{\frac{7}{2}}} + \dots$$

2. Expand $\dfrac{1}{(x+y)^2} = (x+y)^{-2}$.

$$\textit{Ans. } \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} + \frac{5y^4}{x^6} - \ldots$$

3. Expand $\sqrt[3]{1-x} = (1-x)^{\frac{1}{3}}$.

$$\textit{Ans. } 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \frac{10}{243}x^4 - \ldots$$

4. Expand $\sqrt{a^2 - b^2}$.

$$\textit{Ans. } a - \frac{b^2}{2a} - \frac{b^4}{8a^3} - \frac{b^6}{16a^5} - \ldots$$

5. Expand $\sqrt[3]{\dfrac{a^2}{(a^2+b^2)^2}} = \dfrac{a^{\frac{2}{3}}}{(a^2+b^2)^{\frac{2}{3}}} = a^{\frac{2}{3}}(a^2+b^2)^{-\frac{2}{3}}$.

$$\textit{Ans. } \frac{1}{a^{\frac{2}{3}}} - \frac{2b^2}{3a^{\frac{8}{3}}} + \frac{5b^4}{9a^{\frac{14}{3}}} - \frac{40b^6}{81a^{\frac{20}{3}}} + \ldots$$

DIFFERENTIAL METHOD OF SERIES.

356. Definitions.

1. The first order of differences of a series is the series obtained by subtracting the first term of the given series from the second, the second from the third, and so on.

2. The second, third, etc., orders are the series obtained from the first, second, etc., orders in the same manner as the first order is obtained from the given series.

Thus, if the given series be 1, 4, 9, 16, 25, . . .

The 1*st* order of differences is 3, 5, 7, 9, . . .

The 2*d* order of differences is 2, 2, 2, . . .

The 3*d* order of differences is 0, 0, . . .

357. Problem I.

To find the first term of any order of differences and any term of the series.

The series and orders of differences will be as follows:

Series, $t_1, t_2, t_3, t_4, t_5, \ldots t_n, \ldots$

1*st* order, $t_2 - t_1,\ t_3 - t_2,\ t_4 - t_3,\ t_5 - t_4, \ldots$

2*d* order, $t_3 - 2t_2 + t_1,\ t_4 - 2t_3 + t_2,\ t_5 - 2t_4 + t_3, \ldots$

3*d* order, $t_4 - 3t_3 + 3t_2 - t_1,\ t_5 - 3t_4 + 3t_3 - t_2, \ldots$

4*th* order, $t_5 - 4t_4 + 6t_3 - 4t_2 + t_1, \ldots$

. . . .

Let the 1*st* terms of these successive orders be denoted by $d_1, d_2, d_3, d_4, \ldots d_n, \ldots$ Then,

$$d_1 = -t_1 + t_2, \qquad \therefore\ t_2 = t_1 + d_1.$$

$$d_2 = t_1 - 2t_2 + t_3, \qquad \therefore\ t_3 = t_1 + 2d_1 + d_2.$$

$$d_3 = -t_1 + 3t_2 - 3t_3 + t_4, \quad \therefore\ t_4 = t_1 + 3d_1 + 3d_2 + d_3.$$

. . . .

$$(1) \quad d_n = \mp t_1 \pm nt_2 \mp \frac{n(n-1)}{1 \cdot 2} t_3 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} t_4 \mp \ldots$$

The upper sign applies if n is odd; the lower, if n is even.

$$(2) \quad t_n = t_1 + (n-1)\,d_1 + \frac{(n-1)\,(n-2)}{1 \,.\, 2}\,d_2 + \frac{(n-1)\,(n-2)\,(n-3)}{1 \,.\, 2 \,.\, 3}\,d_3 + \ldots$$

358. Examples.

1. Find $d_1, d_2, d_3, \ldots$ and t_n of the series $1^2, 2^2, 3^2, \ldots$

Ans. $d_1 = 3,\ d_2 = 2,\ d_3 = 0,\ t_n = n^2.$

2. Find $d_1, d_2, d_3, \ldots$ and t_n of the series $1^3, 2^3, 3^3, \ldots$

Ans. $d_1 = 7,\ d_2 = 12,\ d_3 = 6,\ d_4 = 0,\ t_n = n^3.$

3. Find $d_1, d_2, d_3, \ldots$ and t_n of the series $1, 3, 6, 10, \ldots$

Ans. $d_1 = 2,\ d_2 = 1,\ d_3 = 0,\ t_n = \dfrac{n\,(n+1)}{2}.$

4. Find $d_1, d_2, d_3, \ldots$ and t_n of the series $1.2.3,\ 2.3.4,\ 3.4.5, \ldots$

Ans. $d_1 = 18,\ d_2 = 18,\ d_3 = 6,\ d_4 = 0,\ t_n = n\,(n^2 + 3n + 2).$

5. Find $d_1, d_2, d_3, \ldots$ and t_n of the series $1\,(m+1),\ 2\,(m+2),\ 3\,(m+3), \ldots$

Ans. $d_1 = m + 3,\ d_2 = 2,\ d_3 = 0,\ t_n = n\,(m + n).$

359. Problem II.

To find the sum of n *terms of the series.*

Let the series be

$$(1) \quad t_1, t_2, t_3, t_4, \ldots t_n, \ldots$$

Let us assume the series

$$(2) \quad 0,\ t_1,\ t_1 + t_2,\ t_1 + t_2 + t_3,\ t_1 + t_2 + t_3 + t_4 + \ldots$$

It is evident that the $(n+1)^{st}$ term of (2) is equal to the sum of n terms of (1), and that t_1 of (1) $= d_1$ of (2), d_1 of (1) $= d_2$ of (2), d_2 of (1) $= d_3$ of (2), ...

But the $(n+1)^{st}$ term of (2), which is the sum of n terms of (1), can be found from formula (2) of the preceding problem, if we substitute $n+1$ for n, 0 for t_1, t_1 for d_1, d_1 of (1) for d_2, d_2 of (1) for d_3, ... and s for t_{n+1}, which give

$$(3) \quad s = nt_1 + \frac{n(n-1)}{1 \,.\, 2} d_1 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} d_2 + \ldots$$

360. Examples.

1. Find the sum of n terms of the series 1, 3, 5, 7, 9, ...,

Ans. $s = n^2$.

2. Find the sum of n terms of the series 1, 2, 3, 4, 5, ...

Ans. $s = \dfrac{n(n+1)}{2}$.

3. Find the sum of n terms of the series $1^3, 2^3, 3^3, 4^3, \ldots$

Ans. $s = \dfrac{n^2(n+1)^2}{4}$.

4. Find the sum of n terms of the series $1^4, 2^4, 3^4, 4^4, \ldots$

Ans. $s = \dfrac{n^5}{5} + \dfrac{n^4}{2} + \dfrac{n^3}{3} - \dfrac{n}{30}$.

5. Find the sum of n terms of the series 1.2, 2.3, 3.4, 4.5, ...

Ans. $\dfrac{n(n+1)(n+2)}{3}$.

6. Find the sum of n terms of the series 1.2.3, 2.3.4, 3.4.5, 4.5.6, ...

Ans. $s = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

APPLICATION TO PILES OF BALLS.

361. Problem I.

To find the number of balls in a triangular pile.

In the 1*st* course the number of balls $=1$.

" " 2*d* " " " " " $=1+2=3$.

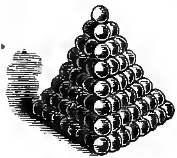
" " 3*d* " " " " " $=1+2+3=6$.

" " 4*th* " " " " " $=1+2+3+4=10$.

. . . .

" " n*th* " " " " " $=1+2+3+\ldots+n$

$$=\frac{n(n+1)}{2}.$$



Hence, the number of balls in a triangular pile of n courses is equal to the sum of n terms of the series 1, 3, 6, 10, . . . $\frac{n(n+1)}{2}$. By formula (3), Art. 359, we find

$$s=\frac{n(n+1)(n+2)}{1\ .\ 2\ .\ 3}.$$

362. Problem II.

To find the number of balls in a square pile.

In the 1*st* course the number of balls $=1$.

" " 2*d* " " " " " $=2^2=4$.

" " 3*d* " " " " " $=3^2=9$.

" " 4*th* " " " " " $=4^2=16$.

. . . .

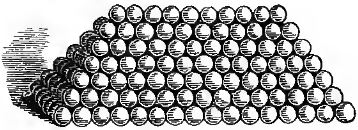
" " n*th* " " " " " $=n^2$.

Hence, the number of balls in a square pile of n courses is equal to the sum of n terms of the series 1, 4, 9, 16, . . . n^2. By formula (3), Art. 359, we find

$$s = \frac{n\,(n+1)\,(2n+1)}{1\ .\ 2\ .\ 3}.$$

363. Problem III.

To find the number of balls in an oblong pile.



Let n = the number of courses = the number of balls in breadth of base.

Let m = the number of balls — 1 in the top row = the number of balls in the length of base — the number of balls in the breadth of base.

The rectangular pile = a square pile of n courses + m triangular strata, each $= \frac{n\,(n+1)}{2}$.

But the number of balls in the square pile is

$$\frac{n\,(n+1)\,(2n+1)}{1\ .\ 2\ .\ 3},$$

and the number of balls in the m triangular strata is

$$\frac{mn\,(n+1)}{2}.$$

Adding and reducing, we have for the number of balls in an oblong pile of n courses

$$s = \frac{n(n+1)(1+2n+3m)}{1 \cdot 2 \cdot 3}.$$

364. Examples.

1. Find the number of balls in a triangular pile of 12 courses. *Ans.* 364.

2. How many balls in a triangular pile of 15 courses, and how many will remain after 5 courses are removed? *Ans.* 680, and 645.

3. In an incomplete triangular pile, the number of balls on one side of the lower base is 20, and on one side of the upper base 10; how many balls in the pile? *Ans.* 1375.

4. How many balls in a square pile of 12 courses? *Ans.* 650.

5. How many balls in a square pile of 15 courses, and how many will remain after 6 courses are removed? *Ans.* 1240, and 1149.

6. In an incomplete square pile, the number of balls on one side of the lower base is 30, and on one side of the upper base 15; how many balls in the pile? *Ans.* 8440.

7. How many balls in a complete oblong pile of which the length of the base is 45, the breadth 15? *Ans.* 4840.

8. In an incomplete oblong pile, the length and breadth of the lower base are respectively 50 and 25, the length and breadth of the upper base 35 and 10; how many balls in the pile? *Ans.* 12240.

9. The number of balls in a triangular pile is to the number in a square pile of the same number of courses as 9 to 17. How many balls in each pile?

Ans. 2925, and 5525.

LOGARITHMS.

365. Definition and Principles.

1. **A logarithm** of a number is the exponent denoting the power to which a fixed number, called the base, must be raised in order to produce the given number. Thus, in the equation $a^x = n$, $x = \log_a n$, which is read $x =$ the logarithm of n to the base a.

2. Any positive number, except 1, may be assumed as the base; but when assumed, it remains fixed for a system of logarithms.

3. There may be an infinite number of systems of logarithms, since there may be an infinite number of bases.

4. The logarithm is dependent on both the base and the number, since a change of either involves a change of the logarithm.

366. General Properties of Logarithms.

1. *In any system, the logarithm of* 1 *is* 0.

For, $a^0 = 1$; $\therefore$ by def., $\log_a 1 = 0$.

2. *In any system, the logarithm of the base is* 1.

For, $a^1 = a$; $\therefore$ by def., $\log_a a = 1$.

3. *In a system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.

For, $a^{-\infty} = \dfrac{1}{a^\infty} = 0$, if $a > 1$; $\therefore$ by def., $\log_{a>1} 0 = -\infty$.

4. *In a system whose base is less than* 1, *the logarithm of* 0 *is* $+\infty$.

For, $a^\infty = 0$, if $a < 1$; $\therefore$ by def., $\log_{a<1} 0 = +\infty$.

367. Brigg's Logarithms.

Brigg's Logarithms, so called from their inventor, are the logarithms of numbers in the system whose base is 10. This is the common system, and it possesses advantages for computation superior to all other systems.

368. Logarithms of the Powers of 10.

$$
\begin{array}{lll}
10^0 = 1; & \therefore & \log_{10} 1 = 0. \\
10^1 = 10; & \therefore & \log_{10} 10 = 1. \\
10^2 = 100; & \therefore & \log_{10} 100 = 2. \\
10^3 = 1000; & \therefore & \log_{10} 1000 = 3. \\
\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Hence, the logarithm of an exact power of 10 is a whole number, equal to the exponent of the power.

Cor. 1. If $n > 1$ and < 10, $\log_{10} n > 0$ and < 1, or $0 +$ a decimal.

Cor. 2. If $n > 10$ and < 100, $\log_{10} n > 1$ and < 2, or $1 +$ a decimal.

Cor. 3. If $n > 100$ and < 1000, $\log_{10} n > 2$ and < 3, or $2 +$ a decimal.

....

Hence, if the number is not an exact power of 10, its logarithm, in the common system, will consist of two parts, an entire part and a decimal part.

The entire part is called the *characteristic*, and the decimal part the *mantissa*.

369. Problem.

To find the laws for the characteristic.

Take the equations

$$(1) \quad 10^x = n.$$

$$(2) \quad 10^1 = 10.$$

$$(1) \div (2) = (3) \quad 10^{x-1} = \frac{n}{10}.$$

$$\therefore \quad \log_{10} \frac{n}{10} = x - 1 = \log_{10} n - 1.$$

Hence, the logarithm of the quotient of any number by 10 is less by 1 than the logarithm of the number.

Let us now take the number 8979, and its logarithm 3.953228, as given in a table of logarithms, and divide the number successively by 10, and for each division subtract

1 from the logarithm of the dividend, and we shall have the following:

$$
\begin{array}{lrl}
\text{Log.} & 8979 & = 3.953228. \\
\text{"} & 897.9 & = 2.953228. \\
\text{"} & 89.79 & = 1.953228. \\
\text{"} & 8.979 & = 0.953228. \\
\text{"} & .8979 & = \bar{1}.953228. \\
\text{"} & .08979 & = \bar{2}.953228. \\
\text{"} & .008979 & = \bar{3}.953228. \\
 & \ldots\ldots\ldots & \ldots\ldots\ldots
\end{array}
$$

The minus sign applies only to the characteristic over which it is placed. The mantissa is always positive, and is the same for all positions of the decimal point.

An inspection of the above will reveal the following laws:

1. *If the number is integral or mixed, the characteristic is positive and is one less than the number of integral figures.*

2. *If the number is entirely decimal, the characteristic is negative and is one greater, numerically, than the number of 0's immediately following the decimal point.*

370. Propositions and Rules.

1. *The logarithm of the product of two numbers is equal to the sum of their logarithms.*

Let (1) $a^x = m$; then, by def., $\log_a m = x$;

and (2) $a^y = n$; then, by def., $\log_a n = y$.

(1) $\times$ (2) $=$ (3) $a^{x+y} = mn$, $\therefore$ by def., $\log_a mn = x + y$.

$$\therefore \quad \log_a mn = \log_a m + \log_a n.$$

Hence, to multiply by means of logarithms, find the logarithms of the factors and take their sum, which will be the logarithm of the product; find the number corresponding, which will be their product.

2. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Let (1) $a^x = m$; then, by def., $\log_a m = x$;

and (2) $a^y = n$; then, by def., $\log_a n = y$.

(1) ÷ (2) = (3) $a^{x-y} = \frac{m}{n}$; then, by def., $\log_a \frac{m}{n} = x - y$.

$$\therefore \quad \log_a \frac{m}{n} = \log_a m - \log_a n.$$

Hence, to divide by means of logarithms, find the logarithms of the numbers, subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient; then find the number corresponding, which will be the quotient.

3. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Let (1) $a^x = n$; then, by def., $\log_a n = x$.

$(1)^p = (2)$ $a^{px} = n^p$; then, by def., $\log_a n^p = px$.

$$\therefore \quad \log_a n^p = p\,(\log_a n).$$

Hence, to raise a number to a given power by means of logarithms, find the logarithm of the number and multiply it by the exponent of the power, and the product will be the logarithm of the power; find the number corresponding, which will be the power.

4. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

Let (1) $a^x = n$; then, by def., $\log_a n = x$.

$\sqrt[r]{(1)} = (2)$ $a^{\frac{x}{r}} = \sqrt[r]{n}$; then, by def., $\log_a \sqrt[r]{n} = \frac{x}{r}$.

$$\therefore \quad \log_a \sqrt[r]{n} = \frac{\log_a n}{r}.$$

Hence, to extract a given root of a number by means of logarithms, find the logarithm of the number, divide it by the index of the root, and the quotient will be the logarithm of the root; then find the number corresponding, which will be the root.

5. *The logarithm of a number to the base* b *is equal to the logarithm of the same number to the base* a, *divided by the logarithm of* b *to the base* a.

Let (1) $a^x = n$; then, by def., $\log_a n = x$;

and (2) $b^y = n$; then, by def., $\log_b n = y$.

Hence, $a^x = b^y$, $\therefore$ $a^{\frac{x}{y}} = b$; then, by def., $\log_a b = \frac{x}{y}$.

$$\therefore \quad \log_a b = \frac{\log_a n}{\log_b n}, \text{ or } \log_b n \times \log_a b = \log_a n.$$

$$\therefore \quad \log_b n = \frac{\log_a n}{\log_a b}.$$

Hence, to find the logarithm of a number corresponding to a given base, divide the logarithm of the number to any base by the logarithm of the given base to the same base.

371. Table of Logarithms from 1 to 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

372. Examples.

1. Multiply 8 by 7 by means of logarithms.

2. Multiply 9 by 6 by means of logarithms.

3. Divide 99 by 9 by means of logarithms.

4. Divide 35 by 7 by means of logarithms.

5. Square 6 by means of logarithms.

6. Cube 4 by means of logarithms.

7. Extract the square root of 81 by means of logarithms.

8. Extract the cube root of 27 by means of logarithms.

9. Find the logarithm of 9 in a system whose base is 8. *Ans.* 1.05664.

10. Find the logarithm of 100 in a system whose base is 15. *Ans.* 1.70055.

11. Find the logarithm of $\frac{abc}{mn}$.

Ans. $\log a + \log b + \log c - \log m - \log n$.

12. Find the logarithm of $\frac{a^m b^n c^p}{d^q}$.

Ans. $m \log a + n \log b + p \log c - q \log d$.

13. Find the logarithm of $a^2 - b^2$.

Ans. $\log (a + b) + \log (a - b)$.

14. Find the logarithm of $\frac{\sqrt{a^2 - b^2}}{(a+b)^2}$.

Ans. $\frac{1}{2} \log (a - b) - \frac{3}{2} \log (a + b)$.

15. Given $100^x = 80$, to find x. *Ans.* $x = 0.951545$.

16. Given $(\sqrt{5})^x = 10$, to find x. *Ans.* $x = 2.861353$.

373. Logarithmic Series.

Let us endeavor, by the method of indeterminate co-efficients, to develop $\log x$, to any base, into a series arranged according to the ascending powers of x.

Let us assume the series,

$$(1) \quad \log x = M + Nx + Px^2 + Qx^3 + \ldots$$

The co-efficients $M, N, P, Q, \ldots$ are independent of x and dependent on the base.

If $x=0$, $\log 0 = M$; but $\log 0 = \mp\infty$, $\therefore\ \mp\infty = M$, a finite quantity, which is absurd; hence, the assumed series will not give the development required.

Let us assume the series,

$$(2) \quad \log x = Mx + Nx^2 + Px^3 + Qx^4 + \ldots$$

If $x=0$, $\log 0 = 0$; but $\log 0 = \mp\infty$, $\therefore\ \mp\infty = 0$, which is absurd; hence, the assumed series will not give the development.

Let us assume the series,

$$(3) \quad \log(1+x) = Mx + Nx^2 + Px^3 + Qx^4 + \ldots$$

If $x=0$, $\log 1 = 0$, which is true.

Let us also assume,

$$(4) \quad \log(1+y) = My + Ny^2 + Py^3 + Qy^4 + \ldots$$

Subtracting (4) from (3), we have

$$(5) \quad \log(1+x) - \log(1+y) = M(x-y) \\ + N(x^2-y^2) + P(x^3-y^3) + \ldots$$

But $\log(1+x) - \log(1+y) = \log\left(\frac{1+x}{1+y}\right) = \log\left(1+\frac{x-y}{1+y}\right)$

$$= M\left(\frac{x-y}{1+y}\right) + N\left(\frac{x-y}{1+y}\right)^2 + P\left(\frac{x-y}{1+y}\right)^3 + Q\left(\frac{x-y}{1+y}\right)^4 + \ldots$$

$$\therefore \quad (6) \quad M\left(\frac{x-y}{1+y}\right) + N\left(\frac{x-y}{1+y}\right)^2 + P\left(\frac{x-y}{1+y}\right)^3 \\ + Q\left(\frac{x-y}{1+y}\right)^4 + \ldots = M(x-y) + N(x^2-y^2) \\ + P(x^3-y^3) + Q(x^4-y^4) + \ldots$$

Dividing (6) by $x - y$, we have

$$(7)\quad M\left(\frac{1}{1+y}\right) + N\frac{x-y}{(1+y)^2} + P\frac{(x-y)^2}{(1+y)^3} + Q\frac{(x-y)^3}{(1+y)^4} + \ldots$$
$$= M + N(x+y) + P(x^2+xy+y^2) + Q(x^3+x^2y+xy^2+y^3) + \ldots$$

Making $y = x$, we have

$$(8)\quad M\left(\frac{1}{1+x}\right) = M + 2Nx + 3Px^2 + 4Qx^3 + \ldots$$

Clearing of fractions and transposing M, we have

$$(9)\quad 0 = \begin{array}{c|c|c|l} M + 2N & x + 3P & x^2 + 4Q & x^3 + \ldots \\ -M + M & +2N & +3P & \end{array}$$

Then, from the principle of indeterminate co-efficients,

$$\begin{aligned} M - M &= 0, \quad \therefore \quad M = M. \\ 2N + M &= 0, \quad \therefore \quad N = -\tfrac{1}{2}M. \\ 3P + 2N &= 0, \quad \therefore \quad P = -\tfrac{2}{3}N = \tfrac{1}{3}M. \\ 4Q + 3P &= 0, \quad \therefore \quad Q = -\tfrac{3}{4}P = -\tfrac{1}{4}M. \\ \ldots\ldots\ldots & \qquad\qquad \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Substituting the values of $N, P, Q, \ldots$ in (3), we have

$$(10)\quad \log(1+x) = M\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots\right).$$

Equation (10) is called the *logarithmic series.*

Hence, the logarithm of a number is the product of two factors; one a series which is dependent on the number, the other independent of the number, and therefore dependent on the base; for, since the logarithm varies with the base, if the base changes, x *remaining the same,* M *must change.*

374. The Modulus.

The modulus of a system of logarithms is the factor dependent on the base.

If we denote the bases of two systems, respectively, by a and b, and their moduli by M and M', then

$$(1)\quad \log_a (1+x) = M\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots\right).$$

$$(2)\quad \log_b (1+x) = M'\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots\right).$$

$$(1) \div (2) = (3)\quad \frac{\log_a (1+x)}{\log_b (1+x)} = \frac{M}{M'}.$$

$$\therefore\quad (4)\quad \log_a (1+x) : \log_b (1+x) :: M : M'.$$

Hence, the logarithms of the same number in two systems are proportional to the moduli of those systems.

375. The Naperian System.

The Naperian system of logarithms, so called from the name of the inventor, is the system whose modulus is 1.

The base of the Naperian system is usually denoted by e.

If the base of another system be denoted by a and its modulus by M, we shall have, by the preceding Article,

$$(1)\quad \log_a (1+x) : \log_e (1+x) :: M : 1.$$

$$\therefore\quad (2)\quad \log_a (1+x) = \log_e (1+x) \times M.$$

Hence, the logarithm of a number in any system is equal to the Naperian logarithm of the same number, multiplied by the modulus of that system.

Dividing (2) by M, and changing the members, we have

$$(3)\quad \log_e(1+x) = \frac{\log_a(1+x)}{M}.$$

Hence, the Naperian logarithm of any number is equal to the logarithm of the same number in any other system, divided by the modulus of that system.

Dividing (2) by $\log_e(1+x)$, and changing the members, we have

$$(4)\quad M = \frac{\log_a(1+x)}{\log_e(1+x)}.$$

Hence, the modulus of any system is equal to the logarithm of any number in that system, divided by the Naperian logarithm of the same number.

376. Problem.

To transform the logarithmic series into a converging series.

Since, in the Naperian system the modulus is 1 and the base e, the logarithmic series in this system will become

$$(1)\quad \log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots$$

Substituting $-x$ for x, (1) becomes

$$(2)\quad \log_e(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \dots$$

Subtracting (2) from (1), we have

$$(3)\quad \log_e(1+x) - \log_e(1-x) = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \dots).$$

But $\log_e(1+x) - \log_e(1-x) = \log_e\left(\frac{1+x}{1-x}\right)$; hence,

(4) $\log_e\left(\frac{1+x}{1-x}\right) = 2\,(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots)$.

Let $\frac{1+x}{1-x} = \frac{y+1}{y}$, then $x = \frac{1}{2y+1}$, and

$$\log_e\left(\frac{1+x}{1-x}\right) = \log_e\left(\frac{y+1}{y}\right) = \log_e(y+1) - \log_e y.$$

$\therefore$ (4) becomes

(5) $\log_e(y+1) - \log_e y$

$$= 2\left[\frac{1}{2y+1} + \frac{1}{3\,(2y+1)^3} + \frac{1}{5\,(2y+1)^5} + \ldots\right].$$

This series is converging for all values of $y > 0$; and the greater the value of y, the more rapidly will the series converge, and the less will be the number of terms necessary to be taken to give the approximation to the required degree of accuracy.

377. Problem.

To compute a Table of Naperian logarithms.

Formula (5) of the preceding Article, by transposing $-\log_e y$, becomes

(1) $\log_e(y+1)$

$$= \log_e y + 2\left[\frac{1}{2y+1} + \frac{1}{3\,(2y+1)^3} + \frac{1}{5\,(2y+1)^5} + \ldots\right].$$

Remembering that $\log_e 1 = 0$, that the logarithm of the product of two numbers is equal to the sum of their logarithms, and that the logarithm of any power of a number

is equal to the logarithm of the number multiplied by the exponent of the power, we shall have, by making in (1) $y = 1, 2, 3, \ldots$ in succession,

$$\log_e\ 1 = 0.000000.$$

$$\log_e\ 2 = 0.000000 + 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \ldots\right)$$
$$= 0.693147.$$

$$\log_e\ 3 = 0.693147 + 2\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \ldots\right)$$
$$= 1.098612.$$

$$\log_e\ 4 = 2 \times \log_e 2 = 1.386294.$$

$$\log_e\ 5 = 1.386294 + 2\left(\frac{1}{9} + \frac{1}{3.9^3} + \frac{1}{5.9^5} + \ldots\right)$$
$$= 1.609438.$$

$$\log_e\ 6 = \log_e 2 + \log_e 3 = 1.791759.$$

$$\log_e\ 7 = 1.791759 + 2\left(\frac{1}{13} + \frac{1}{3.(13)^3} + \frac{1}{5.(13)^5} + \ldots\right)$$
$$= 1.945910.$$

$$\log_e\ 8 = 3 \times \log_e 2 = 2.079442.$$

$$\log_e\ 9 = 2 \log_e 3 = 2.197225.$$

$$\log_e 10 = \log_e 2 + \log_e 5 = 2.302585.$$

. . . .

378. Problem.

To find the modulus of the common system.

By Art. 375, formula (4), we have

$$M = \frac{\log_{10} 10}{\log_e 10} = \frac{1}{2.302585} = 0.4342944819.$$

379. Problem.

To compute the Table of common logarithms.

By Art. 375, formula (2), we have

$$\log_{10}(1+x) = \log_e(1+x) \times M.$$

Hence, to compute common logarithms, multiply the Naperian logarithms by .4342944819, *the modulus of the common system.*

380. Problem.

To find the numerical value of e, *the Naperian base.*

$$\log_{10} e = \log_e e \times M = 1 \times .4342944819 = .4342944819.$$

From a table of common logarithms, we find

$$e = 2.71828128\ldots$$

381. Propositions.

1. *If the base is greater than* 1, *and the number greater than* 1, *the logarithm is positive.*

For, let $a^x = n$, then $x = \log_a n$, in which $a > 1$ and $n > 1$. Then, if x is negative, we have $\frac{1}{a^x} = n$; but $\frac{1}{a^x} < 1$ and $n > 1$; hence, we have a quantity less than 1 equal to a quantity greater than 1, which is absurd. $\therefore$ x can not be negative; $\therefore$ x is positive.

2. *If the base is greater than 1, and the number less than 1, the logarithm is negative.*

For, let $a^x = n$, then $x = \log_a n$, in which $a > 1$ and $n < 1$. Then, if x is positive, $a^x > 1$; but $n < 1$; hence, we have a quantity greater than 1 equal to a quantity less than 1, which is absurd. $\therefore$ x can not be positive; $\therefore$ x is negative.

3. *If the base is less than 1, and the number greater than 1, the logarithm is negative.*

For, let $a^x = n$, then $x = \log_a n$, in which $a < 1$ and $n > 1$. Then, if x is positive, $a^x < 1$; but $n > 1$; hence, we have a quantity less than 1 equal to a quantity greater than 1, which is absurd. $\therefore$ x can not be positive; $\therefore$ x is negative.

4. *If the base is less than 1, and the number less than 1, the logarithm is positive.*

For, let $a^x = n$, then $x = \log_a n$, in which $a < 1$ and $n < 1$. Then, if x is negative, we have $\dfrac{1}{a^x} = n$; but $\dfrac{1}{a^x} > 1$ and $n < 1$; hence, we have a quantity greater than 1 equal to a quantity less than 1, which is absurd. $\therefore$ x can not be negative; $\therefore$ x is positive.

From Props. 1, 2, 3, 4 we have

$$\text{If } x = \log_a n, \text{ then } \begin{cases} 1. \text{ If } a > 1 \text{ and } n > 1,\ x > 0. \\ 2. \text{ If } a > 1 \text{ and } n < 1,\ x < 0. \\ 3. \text{ If } a < 1 \text{ and } n > 1,\ x < 0. \\ 4. \text{ If } a < 1 \text{ and } n < 1,\ x > 0. \end{cases}$$

$\therefore$ {

1st. If the base and the number are both greater or both less than 1, the logarithm is positive.

2d. If the base is greater than 1, and the number is less than 1, or the reverse, the logarithm is negative.

5. *If the base is greater than 1, the modulus is positive; but if the base is less than 1, the modulus is negative.*

For, by Art. 375, formula (4), we have

$$M = \frac{\log_a n}{\log_e n}.$$

$$\text{Then} \begin{cases} \text{If } a > 1 \text{ and } \begin{cases} n > 1 \begin{cases} \log_a n > 0. \\ \log_e n > 0. \end{cases} \\ n < 1 \begin{cases} \log_a n < 0. \\ \log_e n < 0. \end{cases} \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \therefore \quad M > 0. \\ \text{If } a < 1 \text{ and } \begin{cases} n > 1 \begin{cases} \log_a n < 0. \\ \log_e n > 0. \end{cases} \\ n < 1 \begin{cases} \log_a n > 0. \\ \log_e n < 0. \end{cases} \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \therefore \quad M < 0. \end{cases}$$

382. Examples.

1. Given $(10)^{x^2} = 50$, to find x. *Ans.* $x = 1.3034$.

2. Given $(10)^{2x} - 2(10)^x = 24$, to find x.
Ans. $x = .778151$.

3. Given $a^{2x} + 2pa^x = q$, to find x.

$$\textit{Ans.}\ x = \frac{\log\left(-p \pm \sqrt{q + p^2}\right)}{\log a}.$$

4. Given $a^{px}b^{qx}=r$, to find x.

$$Ans.\ x=\frac{\log r}{p\log a+q\log b}.$$

5. Given $x^y=y^x$ and $x^m=y^n$, to find x and y.

$$Ans.\ x=\left(\frac{m}{n}\right)^{\frac{n}{m-n}},\ y=\left(\frac{m}{n}\right)^{\frac{m}{m-n}}.$$

6. Given $a^{2(x^2+2px)}-2ba^{x^2+2px}=c^2$, to find x.

$$Ans.\ x=-p\pm\sqrt{\frac{\log\left(b\pm\sqrt{b^2+c^2}\right)}{\log a}+p^2}.$$

THEORY OF EQUATIONS.

383. Form of the General Equation.

Every equation of the n^{th} degree involving but one unknown quantity may, by clearing of fractions, transposing, reducing, and dividing by the co-efficient of x^n, be placed under the form

$$x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+\ldots+kx+l=0.$$

In this equation, the co-efficients may be integral or fractional, positive or negative, rational or irrational.

The exponents may all be considered positive; for if any of the exponents are negative, let $-p$ be numerically the greatest negative exponent. Then the equation can be freed from negative exponents by multiplying every term by x^p.

384. Problem.

To find the numerical value of the first member, when a particular number is substituted for the unknown quantity.

Take, for example, the cubic equation,

$$x^3 + ax^2 + bx + c = 0.$$

Substituting n for x, the first member becomes

$$n^3 + an^2 + bn + c = 0.$$

When n is a particular number, this result may be most conveniently obtained by detaching the co-efficients, multiplying the co-efficient of the highest power of the unknown quantity by n, adding the product to the next co-efficient, multiplying the sum by n, adding the product to the next co-efficient, and so on. The last sum will be the value sought.

Thus, in the above equation, we shall have

$$\begin{array}{llll}
1, & a, & b, & c. \\
 & n & n^2 + an & n^3 + an^2 + bn \\
\hline
 & n + a, & n^2 + an + b, & n^3 + an^2 + bn + c.
\end{array}$$

The last sum is the result obtained by direct substitution.

385. Illustration.

Let us find the numerical value of the first member of $x^4 + 4x^3 - 5x - 174 = 0$, when 2 is substituted for x.

$$\begin{array}{rrrrr}
1, & +4, & +\ 0, & -\ 5, & -174. \\
 & +2, & +12, & +24, & +\ 38. \\
\hline
 & +6, & +12, & +19, & -136.
\end{array}$$

Hence, when 2 is substituted for x, the first member becomes -136.

Let us now substitute 3 for x.

$$\begin{array}{rrrrr} 1, & +4, & +0, & -5, & -174. \\ & +3, & +21, & +63, & +174. \\ \hline & +7, & +21, & +58, & 0. \end{array}$$

Since 3, when substituted for x, reduces the first member to 0, 3 verifies the equation, and is therefore a root.

386. Examples.

1. Find the numerical values of the first member of the equation $x^4 + 3x^3 - 7x - 16 = 0$, when 2, 3, and 5 are substituted for x. *Ans.* 10, 125, 949.

2. Find the numerical values of the first member of the equation $3x^5 - 2x^4 + 4x^2 - x + 7 = 0$, when 1, 2, and 6 are substituted for x. *Ans.* 11, 85, 20881.

387. Proposition.

If r *is a root of the general equation, the first member is divisible by* x — r.

1ST DEMONSTRATION.

Assuming the division performed, denoting the quotient by q, and the remainder, which is independent of x, by r', we shall have

$$x^n + ax^{n-1} + \ldots + kx + l = (x - r)\,q + r'.$$

Substituting r for x, the first member will become 0, since r is a root, and $(x-r)\,q$ will become 0. Then

$$0 = 0 + r', \quad \therefore \quad r' = 0.$$

Hence, the first member is divisible by $x - r$.

2D DEMONSTRATION.

(1) $x^n + ax^{n-1} + \ldots + kx + l = x^n + ax^{n-1} + \ldots + kx + l.$

(2) $0 = r^n + ar^{n-1} + \ldots + kr + l.$

(1) — (2) = (3) $x^n + ax^{n-1} + \ldots + kx + l$
$= x^n - r^n + a\,(x^{n-1} - r^{n-1}) + \ldots + k\,(x - r).$

But the second member of (3) is divisible by $x - r$; therefore the first member, which is the first member of the general equation, is divisible by $x - r$.

388. Problem.

To find the quotient when the first member of the equation, $x^n + ax^{n-1} + \ldots + kx + l = 0$, *is divided by the unknown quantity, minus a root of the equation.*

Since $x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l$
$= x^n - r^n + a(x^{n-1} - r^{n-1}) + b\,(x^{n-2} - r^{n-2}) + \ldots + k\,(x - r).$

The quotient of the first member divided by $x - r$ can be obtained by dividing the second member by $x - r$, which gives

$$\begin{aligned}
&x^{n-1} + x^{n-2}\,r + x^{n-3}\,r^2 + \ldots + xr^{n-2} + r^{n-1}\\
&+ a\,(x^{n-2} + x^{n-3}\,r + x^{n-4}\,r^2 + \ldots + xr^{n-3} + r^{n-2})\\
&+ b\,(x^{n-3} + x^{n-4}\,r + x^{n-5}\,r^2 + \ldots + xr^{n-4} + r^{n-3})\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+ k.
\end{aligned}$$

Arranging the terms with reference to the descending powers of x, we have

$$x^{n-1} + (r + a)\,x^{n-2} + (r^2 + ar + b)\,x^{n-3} + \ldots$$
$$+ r^{n-1} + ar^{n-2} + br^{n-3} + \ldots + k.$$

The co-efficients of the powers of x in the quotient can evidently be obtained by the process employed in finding the numerical value of the first member, when a particular number is substituted for the unknown quantity. Art. 384.

389. Illustrations.

1. Thus, 3 is a root of the equation,

$$x^4 + 4x^3 - 5x - 174 = 0,$$

the first member is divisible by $x - 3$, and the quotient is obtained thus:

$$\begin{array}{rrrrr} 1, & +4, & +0, & -5, & -174. \\ & +3, & +21, & +63, & +174. \\ \hline & +7, & +21, & +58. & \end{array}$$

$\therefore\quad x^3 + 7x^2 + 21x + 58 =$ the quotient.

$\therefore\quad x^4 + 4x^3 - 5x - 174 = (x-3)\,(x^3 + 7x^2 + 21x + 58).$

2. Again, 2 is a root of the cubic equation

$$x^3 - 9x^2 + 26x - 24 = 0,$$

what are the other roots?

Proceeding as above, we shall have

$$\begin{array}{rrrr} 1, & -9, & +26, & -24. \\ & +2, & -14, & +24. \\ \hline & -7, & +12. & \end{array}$$

$\therefore\quad x^2 - 7x + 12 =$ the quotient.

$\therefore\quad x^3 - 9x^2 + 26x - 24 = (x-2)(x^2 - 7x + 12) = 0.$

$$\therefore \begin{cases} \text{either } x - 2 = 0, & \therefore\quad x = 2. \\ \text{or}\quad x^2 - 7x + 12 = 0, & \therefore\quad x = 4 \text{ or } 3. \end{cases}$$

390. Examples.

1. A root of the equation,

$$x^4 - 12x^3 + 48x^2 - 68x + 15 = 0,$$

is 5; factor the first member.

Ans. $(x-5)(x^3 - 7x^2 + 13x - 3) = 0.$

2. One root of the equation,

$$x^3 - 2x^2 - 23x + 60 = 0,$$

is 4; what are the other roots? *Ans.* 3 and -5.

3. Two roots of the equation,

$$x^4 + 2x^3 - 25x^2 - 26x + 120 = 0,$$

are 4 and -3; what are the other roots? *Ans.* 2, -5.

4. Two roots of the equation,

$$x^4 - 8x^3 - 11x^2 + 198x - 360 = 0,$$

are 3 and -5; what are the other roots? *Ans.* 4, 6.

391. Proposition.

If the first member of the general equation is divisible by x—r, r *is a root of the equation.*

For, let q denote the quotient; then

$$x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = (x-r)\,q.$$

Substituting r for x in both members, we have

$$r^n + ar^{n-1} + br^{n-2} + \ldots + kr + l = 0.$$

Hence, r is a root of the equation.

392. Problem.

To find the remainder when the first member of the general equation is divided by x — p, *if* p *is not a root of the equation.*

Denoting the quotient by q, and the remainder, which is independent of x, by r', we shall have

$$x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = (x - p)\,q + r'.$$

Making $x = p$, we shall have

$$p^n + ap^{n-1} + bp^{n-2} + \ldots + kp + l = r'.$$

Hence, r' is what the first member of the equation becomes when p is substituted for x, which is found by the process of Art. 384.

393. Examples.

1. Find the remainder when the first member of the equation $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$ is divided by $x - 5$. *Ans.* 108.

2. Find the quotient and remainder when the first member of $x^4 - 8x^3 - 11x^2 + 198x - 360 = 0$ is divided by $x - 7$. *Ans.* $x^3 - x^2 - 18x + 72$, 144.

DERIVED FUNCTIONS.

394. Definition and Notation.

A function of a quantity is any expression containing that quantity.

For the sake of brevity, let us denote the equation,

$$x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = 0, \text{ by } f(x) = 0,$$

which is read *a function of* x *is equal to zero.* In this expression, f is not a factor, but a symbol denoting that $f(x)$ is an expression involving x. Let $f(p), f(q), f(r)\ldots$ denote what $f(x)$ becomes when p, q, $r\ldots$ are substituted for x.

395. Problem.

To find the co-efficients of the powers of y, *when* x + y *is substituted for* x *in the equation* f (x) = 0, *and the result is arranged according to the ascending powers of* y.

$$f(x) = x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l.$$

$$f(x+y) = (x+y)^n + a(x+y)^{n-1} + \ldots + k(x+y) + l.$$

Expanding by the Binomial formula, and arranging the result according to the ascending powers of y, $f(x+y)$ becomes

$$x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l + [nx^{n-1} + (n-1)ax^{n-2}$$

$$+ (n-2)bx^{n-3} + \ldots k]y + [n(n-1)x^{n-2}$$

$$+ (n-1)(n-2)ax^{n-3} + (n-2)(n-3)bx^{n-4} + \ldots]\frac{y^2}{1.2}$$

$$+ [n(n-1)(n-2)x^{n-3} + (n-1)(n-2)(n-3)ax^{n-4} + \ldots]\frac{y^3}{1.2.3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ y^n.$$

The co-efficients of y^0 and y^n are $f(x)$ and 1, respectively. Denoting the co-efficients of y, $\frac{y^2}{\underline{|2}}$, $\frac{y^3}{\underline{|3}}$, ... by $f'(x)$, $f''(x)$, $f'''(x)$, ..., respectively, then

$$f(x+y) = f(x) + f'(x)\,y + f''(x)\frac{y^2}{\underline{|2}} + f'''(x)\frac{y^3}{\underline{|3}} + \dots$$

In this development, $f(x)$, the co-efficient of y^0, is the original function of x; $f'(x)$, the co-efficient of y, is the first derived function of $f(x)$; $f''(x)$, the co-efficient of $\frac{y^2}{\underline{|2}}$, is the second derived function of $f(x)$, and so on.

By an inspection of the values of $f(x)$, $f'(x)$, $f''(x)$ ..., we see that any derived function is obtained by multiplying each term of the preceding function by the exponent of x in that term, and diminishing the exponent of x by unity.

Thus, let it be required to find the derived functions of $x^4 + 3x^3 + 5x^2 - 7x - 8 = 0$.

$$\begin{aligned}
f\ (x) &= x^4 + 3x^3 + 5x^2 - 7x - 8. \\
f'\ (x) &= 4x^3 + 9x^2 + 10x - 7. \\
f''\ (x) &= 12x^2 + 18x + 10. \\
f'''\ (x) &= 24x + 18. \\
f''''\ (x) &= 24. \\
f'''''\ (x) &= 0.
\end{aligned}$$

396. Proposition.

In f(x) *involving different powers of* x, *a value sufficiently great may be assigned to* x *as to cause any term which occurs to contain the sum of the terms involving the inferior powers of* x *any finite number of times, and a value sufficiently small may be*

assigned to x *as to cause any term which occurs to contain the sum of the terms involving the superior powers of* x *any finite number of times.*

Let $f(x) = x^n + ax^{n-1} + \ldots + dx^{n-r+1} + \ldots + kx + l$.

1*st*. Let dx^{n-r+1} be the r^{th} term, and let g be the numerical value of the numerically greatest co-efficient of any of the terms involving powers of x inferior to the $(n-r+1)^{th}$ power. The sum of the terms which follow the r^{th} term can not exceed $g(x^{n-r} + x^{n-r-1} + \ldots + x + 1)$ which is equal to $\dfrac{g(x^{n-r+1}-1)}{x-1}$, since the series within the parenthesis is a geometrical progression; but

$$dx^{n-r+1} \div \frac{g(x^{n-r+1}-1)}{x-1} = \frac{d(x-1)x^{n-r+1}}{g(x^{n-r+1}-1)} = \frac{d(x-1)}{g - \dfrac{g}{x^{n-r+1}}}.$$

As x increases the numerator increases, and the denominator approaches the limit g, which it can not exceed. Hence, by making x sufficiently great, the numerator may be made to contain the denominator any finite number of times. Since this fraction expresses the quotient of the r^{th} term divided by the sum of the terms involving the inferior powers of x, and since the r^{th} term may represent any term, any term which occurs, by assigning a value sufficiently great to x, may be made to contain the sum of the terms involving the inferior powers of x any finite number of times.

2*d*. Let $x = \dfrac{1}{y}$; then $f(x)$ becomes

$$\frac{1}{y^n} + \frac{a}{y^{n-1}} + \ldots + \frac{d}{y^{n-r+1}} + \ldots + \frac{k}{y} + l.$$

Reducing to a common denominator and factoring, we have

$$\frac{1}{y^n}(1 + ay + \ldots + dy^{r-1} + \ldots + ky^{n-1} + ly^n).$$

Now, a value sufficiently great may be assigned to y, or, which is the same, a value sufficiently small may be assigned to x, as to cause dy^{r-1} to contain the sum of the terms within the parenthesis, involving the inferior powers of y, any finite number of times, according to the first part of the proposition; hence, $\frac{1}{y^n}$ times this term will contain $\frac{1}{y^n}$ times this sum any finite number of times; that is, dx^{n-r+1} will contain the sum of the terms involving the superior powers of x any finite number of times.

397. Proposition.

If f(s) *and* f(t) *be values of* f(x), *corresponding to the values* x = s *and* x = t, *respectively, then if* x *changes from* s *to* t, *passing through every intermediate value,* f(x) *will change from* f(s) *to* f(t), *and will pass through every intermediate value.*

Let $x = v$, then $f(x) = f(v)$. If in $f(v)$ we substitute $v + h$ for v, we shall have, by Art. 395,

$$f(v + h) = f(v) + f'(v)h + f''(v)\frac{h^2}{\lfloor 2} + f'''(v)\frac{h^3}{\lfloor 3} + \cdots$$

Now, by the preceding proposition, we may make the first term which occurs of the second member of the equation,

$$f(v + h) - f(v) = f'(v)h + f''(v)\frac{h^2}{\lfloor 2} + f'''(v)\frac{h^3}{\lfloor 3} + \ldots,$$

contain the sum of the following terms any finite number of times. Then, by making h small enough, the second member can be made less than any assignable quantity; therefore, $f(v+h)-f(v)$ will be less than any assignable quantity. Hence, as x changes from s to t, passing through every intermediate value, $f(x)$ will change from $f(s)$ to $f(t)$, and will pass through every intermediate value.

Scholium 1. It is not asserted that $f(x)$ always increases or always decreases from $f(s)$ to $f(t)$, but that any two consecutive values of $f(x)$ differ insensibly from each other.

Scholium 2. It is to be observed that s, t, $f(s)$, $f(t)$ are not restricted to positive quantities.

398. Proposition.

If two numbers substituted for x *in the equation,* $\mathrm{f}(\mathrm{x})=0$, *give results with contrary signs, then the equation,* $\mathrm{f}(\mathrm{x})=0$, *has at least one real root between these numbers.*

Let p and t be these numbers; then, by hypothesis, $f(p)$ and $f(t)$ will have contrary signs. But as x changes from p to t, passing through every intermediate value, $f(x)$ will change from $f(p)$ to $f(t)$, and will pass through every intermediate value.

But since $f(p)$ and $f(t)$ have contrary signs, 0 is one of these intermediate values; hence, as x changes from p to t, a certain value of x, which we shall denote by r, will make $f(x)=0$; $\therefore$ r is a root of $f(x)=0$.

399. Proposition.

Any equation of an odd degree, the co-efficients of which are all real, has at least one real root, whose sign is contrary to that of the last term.

Let $f(x) = x^n + ax^{n-1} + \ldots + kx + l = 0$, in which n is odd.

If we substitute for x a value, v, sufficiently great to make x^n greater than the sum of the remaining terms, $f(x)$, which becomes $f(v)$, will have the same sign as v, since n is odd.

If we substitute 0 for x, $f(x)$, which becomes $f(0)$, reduces to l, and, of course, has the same sign as l.

1st. Let l be *negative.*

If $x = 0$, $f(x)$ becomes $f(0) = -l$, a negative quantity. If $x = +v$, $f(x)$ becomes $f(+v)$, a positive quantity.

Hence, the equation $f(x) = 0$ has at least one real root between 0 and $+v$, which root is positive, or has a sign contrary to that of l.

2d. Let l be *positive.*

If $x = 0$, $f(x)$ becomes $f(0) = l$, a positive quantity. If $x = -v$, $f(x)$ becomes $f(-v)$, a negative quantity.

Hence, the equation $f(x) = 0$ has at least one real root between 0 and $-v$, which root is negative, or has a sign contrary to that of l.

400. Proposition.

Any equation of an even degree, the co-efficients of which are all real, and whose last term is negative, has at least two real roots with contrary signs.

Let $f(x) = x^n + ax^{n-1} + \ldots + kx - l = 0$, in which n is even.

Let v be a quantity sufficiently great to make

$$v^n > av^{n-1} + bv^{n-2} + \ldots + kv - l.$$

Then, since n is even, v^n, and consequently $f(v)$, is positive, whether v is positive or negative.

If $x = +v$, $f(x)$ becomes $f(+v)$, a positive quantity. If $x = 0$, $f(x)$ becomes $f(0) = -l$, a negative quantity. If $x = -v$, $f(x)$ becomes $f(-v)$, a positive quantity.

Hence, the equation $f(x) = 0$ has at least one real positive root between $+v$ and 0, and at least one real negative root between 0 and $-v$.

401. Proposition.

If the signs of the terms of the equation, $f(x) = 0$, *are all positive, the equation has no positive root.*

For every positive number substituted for x would make the first member positive, which would not verify the equation.

402. Proposition.

If in the equation, $f(x) = 0$, *the last term and all the terms containing the even powers of* x *have the same sign, and all the terms containing the odd powers of* x *have the contrary sign, the equation has no negative roots.*

For the substitution of a negative quantity for x will make the signs all plus, if the terms containing the even powers of x are plus; otherwise the signs will all become minus.

403. Proposition.

If the equation, $f(x) = 0$, *involves only the even powers of* x, *and all the terms have the same sign, the equation has no real roots.*

For if the signs are all alike, the substitution of any real quantity, positive or negative, for x, will render the signs

of all the terms alike, which will not verify the equation. In this case, therefore, the roots are all imaginary.

404. Proposition.

If the equation, f (x) = 0, *involves only the odd powers of* x, *and the terms all have the same sign, and all contain* x, *the equation has no real root except* x = 0.

For, by factoring, the equation becomes

$$(1) \quad x\,(x^{n-1} + bx^{n-3} + dx^{n-5} + \ldots + k) = 0,$$

which may be satisfied by making $x = 0$, or

$$(2) \quad x^{n-1} + bx^{n-3} + dx^{n-5} + \ldots + k = 0.$$

But equation (2) involves only the even powers of x, and the terms all have the same sign; therefore, it has no real root; hence, equation (1) has no real root except $x = 0$.

405. Remarks on the Existence of a Root.

1. We have found that any equation of an odd degree of the general form, involving only real co-efficients, has at least one real root; and that any equation of an even degree of the general form, involving only real co-efficients, whose last term is negative, has at least two real roots.

2. It now remains to be demonstrated that any equation of the general form has a root.

3. It may, indeed, be granted that every equation which is the statement of a determinate problem has at least one root; for, if not, then the problem is not determinate. But the question is, Has any function of x of the general

form $x^n + ax^{n-1} + \ldots + kx + l$, written at random, and placed equal to 0, a root? It will not do to say, "If the two members of an equation are equal, they must be so for at least *some one* value of the unknown quantity, real or imaginary;" for the members have only been *assumed* equal; and now the question is, Can they be made equal for any value assigned to x?

4. The subjoined demonstration of the proposition, Any equation of the general form has at least one root, is, in substance, Cauchy's; but as it is somewhat difficult, it may be omitted at the discretion of the teacher.

406. Proposition.

Any equation, f (x) = 0, *of the general form, in which the co-efficients are real or imaginary, has at least one root, real or imaginary.*

Let $a + b\sqrt{-1}$ be substituted for x in $f(x)$; then $f(x)$ will become $P + Q\sqrt{-1}$, P denoting the algebraic sum of the real quantities, and $Q\sqrt{-1}$ the algebraic sum of the imaginary quantities. If $P = 0$ and $Q = 0$, then $P + Q\sqrt{-1} = 0$, and $a + b\sqrt{-1}$ is a root of the equation $f(x) = 0$. Let us suppose that P and Q are not both 0. Let, now, $a + b\sqrt{-1} + h$ be substituted for x in $f(x)$, which may be done by substituting $a + b\sqrt{-1}$ for x in the development,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h}{\underline{|2}} + f'''(x)\frac{h}{\underline{|3}} + \cdots$$

Then $f(x)$ will become $P + Q\sqrt{-1}$, as above. Some of the co-efficients may vanish, but all can not vanish, since the last term is h^n.

Let h^p be the lowest power of h, whose co-efficient does not vanish. Denote this co-efficient by $R+S\sqrt{-1}$; then

$$f(a+b\sqrt{-1}+h)=P+Q\sqrt{-1}+(R+S\sqrt{-1})h^p+\ldots$$

Let $P'+Q'\sqrt{-1}=P+Q\sqrt{-1}+(R+S\sqrt{-1})h^p+\ldots$

Let $h=kt$, t^p being -1 or $+1$; then

$$P'+Q'\sqrt{-1}=P+Q\sqrt{-1}\mp(R+S\sqrt{-1})k^p+\ldots$$

$$\therefore \quad \begin{cases} P'=P\mp Rk^p+\ldots \\ Q'=Q\mp Sk^p+\ldots \end{cases}$$

$$\therefore \quad P'^2+Q'^2=P^2+Q^2\mp 2(PR+QS)k^p+\ldots$$

$$\therefore \quad P'^2+Q'^2-P^2-Q^2=\mp 2(PR+QS)k^p+\ldots$$

Now, if $PR+QS$ is not 0, the sign of the second member, by making k sufficiently small, will be the same as that of $\mp 2(PR+QS)k^p$, which may always be made minus by making $t^p=-1$ or $+1$, according as $PR+QS$ is positive or negative.

$\therefore \quad P'^2+Q'^2-P^2-Q^2$ can always be made negative.

$$\therefore \quad P'^2+Q'^2<P^2+Q^2.$$

If $PR+QS$ is 0, let $t^p=-\sqrt{-1}$ or $+\sqrt{-1}$; then

$$P'+Q'\sqrt{-1}=P+Q\sqrt{-1}\mp(R+S\sqrt{-1})k^p\sqrt{-1}+\ldots$$

$$\therefore \quad \begin{cases} P'=P\pm Sk^p+\ldots \\ Q'=Q\mp Rk^p+\ldots \end{cases}$$

$$\therefore \quad P'^2+Q'^2=P^2+Q^2\mp 2(QR-PS)k^p+\ldots$$

$$\therefore \quad P'^2+Q'^2-P^2-Q^2=\mp 2(QR-PS)k^p+\ldots$$

Now, $(PR+QS)^2+(QR-PS)^2=(P^2+Q^2)(R^2+S^2)$.

By hypothesis, neither P^2+Q^2 is 0, nor R^2+S^2 is 0.

$\therefore$ $(PR+QS)^2+(QR-PS)^2$ is not 0.

But $PR+QS$ is 0; $\therefore$ $(PR+QS)^2$ is 0.

$\therefore$ $(QR-PS)^2$ is not 0; $\therefore$ $QR-PS$ is not 0.

The sign of $P'^2+Q'^2-P^2-Q^2$, if k is sufficiently small, is the same as that of $\mp 2(QR-PS)k^p$, which may always be made minus by making $t^p=-\sqrt{-1}$ or $+\sqrt{-1}$, according as $QR-PS$ is positive or negative.

$\therefore$ $P'^2+Q'^2-P^2-Q^2$ can always be made negative.

$\therefore$ $P'^2+Q'^2<P^2+Q^2$.

Hence, by assigning proper values to h, we can always make $P'^2+Q'^2<P^2+Q^2$; that is, if P^2+Q^2 is susceptible of any value however small, $P'^2+Q'^2$ is susceptible of a value still smaller. This smaller value of $P'^2+Q'^2$ may then be attributed to P^2+Q^2, and we can make $P'^2+Q'^2$ smaller than the new value of P^2+Q^2, and so on. Hence, it is possible that finally $P'^2+Q'^2$ become 0; and since P'^2 and Q'^2 are both squares, and therefore positive so long as they are not 0, when $P'^2+Q'^2$ becomes 0, P'^2 must be 0, and Q'^2 must be 0; that is, P' and Q' vanish simultaneously. $\therefore$ $P'+Q'\sqrt{-1}=0$; hence, $a+b\sqrt{-1}+h$, which, substituted for x, makes $f(x)=P'+Q'\sqrt{-1}=0$, is a root of $f(x)=0$.

407. Proposition.

Every equation of the n[th] *degree has* n *roots.*

$$(1) \quad x^n + ax^{n-1} + \ldots + kx + l = 0.$$

Let r be a root of this equation, then $x - r$ is a factor of the first member; then (1) becomes

$$(2) \quad (x - r)\,(x^{n-1} + a'x^{n-2} + \ldots + k'x + l') = 0.$$

Equation (2), and therefore (1), is satisfied if

$$(3) \quad x^{n-1} + a'x^{n-2} + \ldots + k'x + l' = 0.$$

Let s be a root of this equation, then

$$(4) \quad (x - s)\,(x^{n-2} + a''x^{n-3} + \ldots + k''x + l'') = 0.$$

Every binomial factor found reduces the degree of the polynomial factor by 1; hence, after $n - 2$ binomial factors have been found, the polynomial factor will be reduced to one of the second degree, and this factor can be resolved into two binomial factors of the first degree, $x - u$ and $x - v$. Hence, (1) becomes

$$(5) \quad (x - r)\,(x - s) \ldots (x - u)\,(x - v) = 0.$$

Since the equation may be satisfied by making either of these factors equal to 0, which will give one root, the equation has n roots.

The equation has no more than n roots; for, if it has another root, w, $x - w$ would be a factor, and the product of these $n + 1$ factors, placed equal to 0, would be an equation of the $(n + 1)^{st}$ degree.

It does not follow that the n roots are all real and different, for two or more of them may be imaginary or equal.

408. Problem.

To find the relations which exist between the co-efficients and the roots of the equation, f (x) = 0.

1. If there are two roots, r and s, we shall have

$$(x-r)(x-s) = \begin{array}{r|l} x^2 - r & x + rs = 0. \\ - s & \end{array}$$

2. If there are three roots, r, s, and t, we have

$$\begin{array}{r|r|l} x^3 - r & x^2 + rs & x - rst = 0. \\ - s & + rt & \\ - t & + st & \end{array}$$

3. If there are four roots, r, s, t, and u, we have

$$\begin{array}{r|r|r|l} x^4 - r & x^3 + rs & x^2 - rst & x + rstu = 0. \\ - s & + rt & - rsu & \\ - t & + ru & - rtu & \\ - u & + st & - stu & \\ & + su & & \\ & + tu & & \end{array}$$

Thus far, the following laws hold:

1. The number of terms is one more than the number of binomial factors employed.

2. The exponent of x in the first term is equal to the number of binomial factors employed, and decreases by unity in the successive terms to the right.

3. The co-efficient of the first term is unity; the co-efficient of the second term is the sum of the roots with their signs changed; the co-efficient of the third term is the sum of the products of the roots, taken in combinations of two; the co-efficient of the fourth term is the sum of the products of the roots with their signs changed, taken in combinations of three; the last term is the product of the roots, if the equation is of an even degree, or the product of the roots with their signs changed, if the equation is of an odd degree.

409. Laws Generalized.

To prove the above laws general, assume them true for $n-1$ factors, $x-r$, $x-s$, $\ldots x-u$, whose product gives the equation

$$x^{n-1}+a'x^{n-2}+b'x^{n-3}+\ldots+l',$$

in which
$$a'=-r-s-\ldots-u,$$

$$b'=rs+\ldots+ru+\ldots su.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$l'=\pm rs\ldots u.$$

Let us introduce another factor, $x-v$, thus:

$$\begin{array}{l}
x^{n-1}+a'x^{n-2}+b'x^{n-3}+\ldots+l'. \\
x-v \\
\hline
\end{array}$$

$$\begin{array}{r|r|r}
x^{n}+a' & x^{n-1}+\ b' & x^{n-2}+\ldots \qquad\qquad \\
-\ v & -a'v & \qquad\qquad -l'v.
\end{array}$$

1. The same law holds for the number of terms.

2. The same law holds for the exponents.

3. The same law holds for the co-efficients, since

$$a' - v = -r - s - \dots - u - v,$$

$$b - a'v = rs + \dots + ru + \dots + su + \dots + rv + sv + \dots + uv,$$

. . . .

$$-l'v = \mp rs \dots uv.$$

Hence, if the laws hold for $n-1$ factors, they hold for n factors; that is, if they hold for a certain number of factors, they hold for one more. But, by trial, we find that they hold for four factors; hence they hold for five; and if for five, then for six, and so on.

Cor. 1. The last term is divisible by each of the roots.

Cor. 2. If the last term is 0, one of the roots is 0.

410. Remark on the Composition of Equations.

How do we obtain $x^2, x^3, \dots x^n$ by multiplying the binomial factors together, when the value of x in one factor is not the same as its value in the other factors?

Thus, take the factors

$$x - r = 0, \text{ in which } x = r;$$

$$x - s = 0, \text{ in which } x = s.$$

$$\therefore \quad (1) \quad (x-r)(x-s) = x^2 \left.\begin{array}{r} -r \\ -s \end{array}\right| x + rs = 0.$$

To explain how x^2 is obtained when $x = r$ in one factor and $x = s$ in the other, let us suppose that x retains its

proper value in either factor, and that this value be assigned to x in the other factor. One factor will remain 0, but not the other; the product will be 0, and the equation obtained will be true.

Denoting the value of x which is equal to r by x', and the value of x which is equal to s by x'', we have

$$(2) \quad (x' - r)(x'' - s) = 0.$$

Substituting x' for x'', we have

$$(3) \quad (x' - r)(x' - s) = x'^2 \begin{array}{r|l} -r & x' + rs = 0, \\ -s & \end{array}$$

which is true, since $x' - r = 0$.

Substituting x'' for x', we have

$$(4) \quad (x'' - r)(x'' - s) = x''^2 \begin{array}{r|l} -r & x'' + rs = 0, \\ -s & \end{array}$$

which is true, since, $x'' - s = 0$.

But (3) and (4) are evidently involved in (1), since the co-efficients are the same in the three equations. Since (1) is the general equation, involving (3) and (4) as particular cases, it ought, by its different roots, to give $x = r$, found from (3), and $x = s$, found from (4).

411. Examples.

1. Find the equation whose roots are 2, -3, 4.

Ans. $x^3 - 3x^2 - 10x + 24 = 0$.

2. Form the equation whose roots are 3, -2, $1 + \sqrt{2}$, and $1 - \sqrt{2}$. *Ans.* $x^4 - 3x^2 - 5x^2 + 13x + 6 = 0$.

412. Problem.

To find the equal roots of an equation.

$$(1) \quad f(x) = x^n + ax^{n-1} + \ldots + kx + l = 0.$$

Assuming the second member factored, we have

$$(2) \quad f(x) = (x - r)(x - s) \ldots (x - u)(x - v).$$

Substituting $y + x$ for x, we have, Art. 395,

$$(3) \quad f(x) + f'(x)\,y + f''(x)\frac{y}{\underline{|2}} + f'''(x)\frac{y}{\underline{|3}} + \ldots$$
$$= (y + x - r)(y + x - s) \ldots (y + x - u)(y + x - v).$$

The co-efficient of y in the first member is $f'(x)$; the co-efficient of y in the second member is the sum of the products of $x - r$, $x - s$, $\ldots\, x - u$, $x - v$, taken in combinations of $n - 1$ together, and is, therefore,

$$\frac{f(x)}{x - r} + \frac{f(x)}{x - s} + \ldots + \frac{f(x)}{x - u} + \frac{f(x)}{x - v}.$$

But since (3) is an identical equation, the co-efficients of y in the two members must be equal.

$$\therefore \quad (4) \quad f'(x) = \frac{f(x)}{x - r} + \frac{f(x)}{x - s} + \ldots + \frac{f(x)}{x - u} + \frac{f(x)}{x - v}.$$

But if $f(x)$ has, for example, p roots equal to r, and q roots equal to s, and the other roots unequal, we have

$$(5) \quad f(x) = (x - r)^p (x - s)^q \ldots (x - u)(x - v).$$

Dividing $f(x)$ by each of the p factors equal to $x - r$, and by each of the q factors equal to $x - s$, and by each of

the other factors corresponding to the unequal roots, we have p quotients equal to $\frac{f(x)}{x-r}$, q quotients equal to $\frac{f(x)}{x-s}$, and the other quotients, ... $\frac{f(x)}{x-u}$, $\frac{f(x)}{x-v}$; then,

$$(6) \quad f'(x) = \frac{pf(x)}{x-r} + \frac{qf(x)}{x-s} + \ldots + \frac{f(x)}{x-u} + \frac{f(x)}{x-v}.$$

It is evident that $(x-r)^{p-1}(x-s)^{q-1}$ is the g. c. d. of the terms of $f'(x)$ in the second member of (6), that it is a divisor of $f(x)$, and that it is the g. c. d. of $f(x)$ and $f'(x)$.

Conversely, if $(x-r)^{p-1}(x-s)^{q-1}$ is the g. c. d. of $f(x)$ and $f'(x)$, $f(x)=0$ has p roots equal to r, and q roots equal to s. It is also evident that if $f(x)$ and $f'(x)$ have no common divisor except 1, $f(x)$ has no equal roots.

413. Rule.

1. *Find* $f'(x)$, *the first derived function of* $f(x)$.

2. *Find the g. c. d. of* $f(x)$ *and* $f'(x)$.

3. *If the g. c. d. of* $f(x)$ *and* $f'(x)$ *is* 1, $f(x)=0$ *has no equal roots.*

4. *If the g. c. d. of* $f(x)$ *and* $f'(x)$ *is* $(x-r)^{p-1}(x-s)^{q-1}$ ..., $f(x)=0$ *has* p *roots equal to* r, *and* q *roots equal to* s ...

414. Examples.

1. Find the equal roots of $x^4-11x^3+44x^2-76x+48=0$.

SOLUTION.

$$f(x) = x^4 - 11x^3 + 44x^2 - 76x + 48.$$

$$f'(x) = 4x^3 - 33x^2 + 88x - 76.$$

The g. c. d. of $f(x)$ and $f'(x)$ is $x-2$. Hence, $f(x)$ has two roots equal to 2, and is divisible by $(x-2)^2$.

Dividing $f(x)$ by $(x-2)^2$, and placing the quotient equal to 0, we have $x^2-7x+12=0$, which is the equation containing the other roots, which are 3 and 4.

2. Find the equal roots of the equation,

$$x^5-2x^4+3x^3-7x^2+8x-3=0,$$

and the remaining roots.

Ans. 1, 1, 1, $\frac{-1+\sqrt{-11}}{2}$, $\frac{-1-\sqrt{-11}}{2}$.

3. Find all the roots of the equation,

$$x^4-7x^3+9x^2+27x-54=0.$$

Ans. 3, 3, 3, -2.

415. Proposition.

If an equation involving only real co-efficients has imaginary roots, the number of these roots is even.

For, dividing by the factors corresponding to the real roots, the last quotient must be of an even degree; for, if the quotient were of an odd degree, placing it equal to 0, we should have an equation of an odd degree which would have at least one real root. But, by hypothesis, we have divided by all the factors corresponding to the real roots; hence, the polynomial factor, which is the product of the factors corresponding to the imaginary roots, is of an even degree, and has an even number of roots; hence, the number of imaginary roots is even.

416. Proposition.

If the equation, f (x) = 0, *has one imaginary root of the form* $a + b\sqrt{-1}$, *it has another of the form* $a - b\sqrt{-1}$.

Substituting the root $a + b\sqrt{-1}$ for x, the result will consist of two parts: first, the real quantities involving the odd and even powers of a and the even powers of $b\sqrt{-1}$; second, imaginary quantities involving the odd powers of $b\sqrt{-1}$. Let the sum of the real quantities be P, and the sum of the imaginary quantities be $Q\sqrt{-1}$. Then, since $a + b\sqrt{-1}$ is a root of $f(x) = 0$, the result will be $P + Q\sqrt{-1} = 0$. Since P is real and $Q\sqrt{-1}$ is imaginary, they are unequal; and since their sum is 0, each one must be 0; that is, $P = 0$ and $Q\sqrt{-1} = 0$. Now, let $a - b\sqrt{-1}$ be substituted for x. We shall then have the same result as before, except the sign of $Q\sqrt{-1}$; hence, the result of this substitution will be $P - Q\sqrt{-1} = 0$, since $P = 0$ and $Q\sqrt{-1} = 0$; hence, $a - b\sqrt{-1}$ is a root, since it reduces the first member to 0.

417. Proposition.

If the co-efficients of the equation, f (x) = 0, *are all integral, the rational roots are all integral.*

Let $$x^n + ax^{n-1} + \ldots + kx + l = 0,$$

in which all the co-efficients are integral, and let us suppose that one root is equal to the irreducible fraction $\frac{r}{s}$, which is in its lowest terms. Then we shall have

$$\frac{r^n}{s^n} + a\frac{r^{n-1}}{s^{n-1}} + \ldots + k\frac{r}{s} + l = 0.$$

Multiplying by s^{n-1}, and transposing, we have

$$\frac{r^n}{s} = -ar^{n-1} - \ldots - krs^{n-2} - ls^{n-1}.$$

That is, an irreducible fraction is equal to the sum of several integral numbers, which is impossible.

418. Proposition.

If the signs of the alternate terms of a complete equation be changed, the signs of all the roots will be changed.

$$(1) \quad x^n + ax^{n-1} + \ldots + kx + l = 0.$$

Let r be any root of this equation; then

$$(2) \quad r^n + ar^{n-1} + \ldots + kr + l = 0.$$

1*st.* If n is even, we have, by changing the signs of the alternate terms of (1),

$$(3) \quad x^n - ax^{n-1} + \ldots - kx + l = 0.$$

Substituting $-r$ for x, we have

$$(4) \quad r^n + ar^{n-1} + \ldots + kr + l = 0.$$

But (4) is identical with (2); $\therefore$ $-r$ is a root of (3).

2*d.* If n is odd, we have, by changing the signs of the alternate terms of (1),

$$(5) \quad x^n - ax^{n-1} + \ldots + kx - l = 0.$$

Substituting $-r$ for x, and changing all the signs, we have

$$(6) \quad r^n + ar^{n-1} + \ldots + kr + l = 0.$$

But (6) is identical with (2); $\therefore$ $-r$ is a root of (5).

PERMANENCES AND VARIATIONS.

419. Definitions.

1. **A permanence** of signs is the occurrence of two consecutive terms with like signs. Thus, $x + a$, $-x - a$.

2. **A variation** of signs is the occurrence of two consecutive terms with unlike signs. Thus, $x - a$, $-x + a$.

In $x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 44x + 36 = 0$ there are four variations and three permanences.

420. Descartes's Rule for the Signs.

In any equation, the number of positive roots can not exceed the number of variations of signs; and, in a complete equation, the number of negative roots can not exceed the number of permanences of signs.

In the equation, $x - a = 0$, there is one positive root, $+a$, and one variation.

In the equation, $x + a = 0$, there is one negative root, $-a$, and one permanence.

In the equation, $x^2 - (a + b)x + ab = 0$, there are two positive roots, $+a$ and $+b$, and two variations.

In the equation, $x^2 + (a + b)x + ab = 0$, there are two negative roots, $-a$ and $-b$, and two permanences.

We shall now prove that if the degree of the equation is increased by unity by the introduction of a factor of the form $x - v$, corresponding to the positive root $x = v$, the number of variations will be increased by 1.

Let the signs of the terms of the equation before the introduction of the factor $x - v$ be

$$+ + - + - - + +$$

Introducing the new factor $x - v$, corresponding to the positive root $+v$, the signs of the product will be found thus:

$$
\begin{array}{lllllllll}
+ & + & - & + & - & - & + & + & \\
+ & - & & & & & & & \\
\hline
+ & + & - & + & - & - & + & + & \\
 & - & - & + & - & + & + & - & - \\
\hline
+ & \pm & - & + & - & \pm & + & \pm & -
\end{array}
$$

Taking the ambiguous sign either $+$ or $-$, the number of permanences will not be increased; but since there is one more sign, the number of variations is increased by 1; hence, since the introduction of a new factor corresponding to a positive root makes another variation, the number of positive roots can not exceed the number of variations.

If the equation is complete, changing the signs of the alternate terms will change the signs of all the roots, the permanences will become variations, and the variations will become permanences; but, in the resulting equation, the number of positive roots can not exceed the number of variations; hence, in the given equation, the number of negative roots can not exceed the number of permanences.

If the equation is complete, or is made so by the introduction of the missing terms with the co-efficient 0, the whole number of variations and permanences will be equal to the degree of the equation; but the number of positive roots plus the number of negative roots is equal to the degree of the equation. Therefore, the number of variations plus the number of permanences is equal to the number of positive roots plus the number of negative roots.

Calling the number of variations v, the number of permanences p, the number of positive roots p', and the number of negative roots n', we have

$$v + p = p' + n'.$$

Now, p' can not exceed v, nor can p' be less than v; for then n' would exceed p, which is impossible; $\therefore\ p' = v$; $\therefore\ n' = p$. Hence, if the roots are all real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences.

By means of Descartes's Rule, we can often detect the presence of imaginary roots. Thus, take the equation

$$x^2 + 9 = 0.$$

Making the equation complete, we have

$$x^2 \pm 0x + 9 = 0.$$

Now, it is immaterial which sign of $\pm 0x$ is taken. Taking the plus sign, there are no variations; hence, there are no positive roots. Taking the minus sign, there are no permanences; hence, there are no negative roots. Hence, the roots are all imaginary. In fact, $x = \pm 3\sqrt{-1}$.

TRANSFORMATION OF EQUATIONS.

421. First Transformation.

To transform one equation into another, whose roots shall be the same as the roots of the given equation, but with contrary signs.

This transformation is accomplished by substituting $-y$ for x; for then $y = -x$.

422. Examples.

1 Transform $x^2 - 4x = 5$ into another equation, whose roots shall be the same as those of the given equation, but with contrary signs, and solve both equations.

2. Transform $x^2 + 6x = 7$ into another equation, whose roots shall be the same as those of the given equation, but with contrary signs, and solve both equations.

423. Second Transformation.

To transform one equation into another, whose roots shall be the reciprocals of the roots of the given equation.

This transformation is accomplished by substituting $\frac{1}{y}$ for x; for then $y = \frac{1}{x}$.

424. Examples.

1. Transform $x^2 - 8x = 9$ into another equation, whose roots shall be the reciprocals of the roots of the given equation, and solve both equations.

2. Transform $x^2 - 12x = -11$ into another equation, whose roots shall be the reciprocals of the roots of the given equation, and solve both equations.

425. Third Transformation.

To transform one equation into another, whose roots shall be multiples or submultiples of the roots of the given equation.

This transformation is accomplished by substituting $\frac{y}{m}$ or my for x; for then $y = mx$ or $\frac{x}{m}$.

Cor. By this transformation, the co-efficient of the first term may be made 1, and fractions avoided. Thus, let

$$6x^3 - 3x^2 + 2x = 6.$$

Substitute $\frac{y}{6}$ for x; then we shall have

$$\frac{6y^3}{6^3} - \frac{3y^2}{6^2} + \frac{2y}{6} = 6.$$

Multiplying by 6^2, we have

$$y^3 - 3y^2 + 12y = 216.$$

426. Examples.

1. Transform $3x^2 + 4x = 95$ into another equation, whose roots shall be 3 times the roots of the given equation, and solve both equations.

2. Transform $5x^2 + 4x = 273$ into another equation, whose roots shall be $\frac{1}{5}$ of the roots of the given equation, and solve both equations.

3. Transform $3x^3 + \frac{5}{4}x^2 - \frac{2}{3}x + \frac{3}{8} = 0$ into another equation, of which the co-efficients are all entire, and the co-efficient of the first term unity.

Ans. $y^3 + 5y^2 - 32y + 216 = 0.$

427. Fourth Transformation.

To transform one equation into another, whose roots shall be greater or less, by a given quantity, than the roots of the given equation.

Let us transform the equation

$$(1) \quad x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = 0,$$

into another whose roots shall be less by q than the roots of the given equation.

Let $y = x - q$, then $x = y + q$.

Substituting this value of x in (1), developing and arranging with reference to the descending powers of y, the equation will take the form,

$$(2)\quad y^n + a'y^{n-1} + b'y^{n-2} + \ldots + k'y + l' = 0.$$

Substituting $x - q$ for y in (2), we have

$$(3)\quad (x-q)^n + a'(x-q)^{n-1} + \ldots + k'(x-q) + l' = 0.$$

Equation (3), when developed and reduced, must be identical with (1); for (2) was obtained from (1) by substituting $y + q$ for x, and (3) was obtained from (2) by substituting $x - q$ for y, which must reproduce (1).

Our object is to find the co-efficients in (2). If we divide (3) by $x - q$, the quotient will be

$$(4)\quad (x-q)^{n-1} + a'(x-q)^{n-2} + \ldots + k',$$

and the remainder l', the last term of (2).

Dividing (4) by $x - q$, the quotient will be

$$(5)\quad (x-q)^{n-2} + a'(x-q)^{n-3} + \ldots,$$

and the remainder k', the co-efficient of y in (2). In a similar manner, the co-efficients of y^2, $y^3 \ldots$, in (2) can be found. But since (3) and (1) are identical, we can obtain the co-efficients in (2) by operating on (1) as we have on (3), which need not be obtained. The operation of division is performed as in Art. 389, and the remainder is found as in Art. 392.

428. Illustration.

Let us transform the equation,

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0,$$

into another whose roots shall be less than the roots of the given equation by 3.

We are to divide the first member of the equation by $x - 3$, the quotient by $x - 3$, and so on; the remainders will be the co-efficients of the transformed equation.

OPERATION.

$$\begin{array}{rrrrr}
1 - 3 & - 15 & + 49 & - 12 \\
+ 3 & + \ 0 & - 45 & + 12 \\
\hline
+ 0 & - 15 & + \ 4 & + \ 0 \\
+ 3 & + \ 9 & - 18 & \\
\hline
+ 3 & - \ 6 & - 14 & \\
+ 3 & + 18 & & \\
\hline
+ 6 & + 12 & & \\
+ 3 & & & \\
\hline
+ 9 & & &
\end{array}$$

Performing the first division, we have, for the co-efficients of the quotient, 1, + 0, — 15, + 4, and the remainder 0, which is the last term of the transformed equation.

As the co-efficients of the quotient thus obtained are already detached, we proceed with the second division, and obtain — 14 for the second remainder, which is the co-efficient of y in the transformed equation.

In a similar manner, we find that 12 is the co-efficient of y^2, 9 the co-efficient of y^3, and 1 the co-efficient of y^4. Hence, the transformed equation is

$$y^4 + 9y^3 + 12y^2 - 14y + 0 = 0.$$

429. Examples.

1. Transform the equation, $x^3 - 27x - 36 = 0$, into another whose roots are less than the roots of the given equation by 3. *Ans.* $y^3 + 9y^2 - 90 = 0$.

2. Transform $x^4 - 18x^3 - 32x^2 + 17x + 9 = 0$ into an equation whose roots are less by 5 than the roots of the given equation.

Ans. $y^4 + 2y^3 - 152y^2 - 1153y - 2331 = 0.$

3. Transform $x^5 + 2x^3 - 6x^2 - 10x = 0$ into an equation whose roots are less by 2 than the roots of the given equation. *Ans.* $y^5 + 10y^4 + 42y^3 + 86y^2 + 70y + 4 = 0.$

4. Transform $x^3 - 4x^2 - 17x + 60 = 0$ into an equation whose roots are greater by 4 than the roots of the given equation. *Ans.* $y^3 - 16y^2 + 63y = 0.$

430. Fifth Transformation.

To transform one equation into another, whose second term is wanting.

(1) $x^n + ax^{n-1} + \ldots + kx + l = 0.$

$$\text{Let } y = x + \frac{a}{n}, \text{ then } x = y - \frac{a}{n}.$$

Substituting $y - \frac{a}{n}$ for x, we shall have

(2) $\left(y - \frac{a}{n}\right)^n + a\left(y - \frac{a}{n}\right)^{n-1} + \ldots + k\left(y - \frac{a}{n}\right) + l.$

Developing, we shall find that the co-efficient of y^{n-1} is 0, or the second term disappears.

This transformation is accomplished by successive divisions of (1) by $x + \frac{a}{n}$, as in the preceding Article.

431. Examples.

1. Transform $x^3 - 6x^2 + 8x - 2 = 0$ into an equation wanting the second term. *Ans.* $y^3 - 4y - 2 = 0.$

2. Transform $x^3 - 6x^2 + 7x - 2 = 0$ into an equation wanting the second term. *Ans.* $y^3 - 5y - 4 = 0$.

3. Transform the equation

$$x^5 - 6x^4 + 7.4x^3 + 7.92x^2 - 17.872x - .79232 = 0,$$

into another wanting the second term.

Ans. $y^5 - 7y^3 + 2y - 8 = 0$.

LIMITS OF THE ROOTS.

432. Definitions.

1. **The limits** of the roots of an equation are the quantities between which the roots are situated.

2. **A superior limit** of the roots is a quantity greater than any of the roots.

3. **An inferior limit** of the roots is a quantity less than any of the roots.

433. Proposition.

A superior limit of the roots of an equation, or any quantity greater, will, when substituted for the unknown quantity, render the first member positive.

If a superior limit made the first member 0, it would be a root and not a limit. Therefore, a superior limit can not reduce the first member to 0.

If a superior limit made the first member negative, then, since a positive value, greater than the limit, may be assigned to x which will make x^n numerically greater than the sum of all the other terms, in which case the first member is positive, there would be a root included between this value and the superior limit, and this root would be greater than the limit, which is impossible. Therefore, a superior limit renders the first member positive.

If any positive number greater than the superior limit makes the first member negative, then there would be a root included between this number and the superior limit, and this root would be greater than the limit, which is impossible. Therefore, any positive quantity greater than a superior limit will make the first member positive.

434. Proposition.

Unity plus that root of the greatest negative co-efficient, whose index is equal to the number of terms preceding the first negative term, is a superior limit of the positive roots.

Let g be the numerically greatest negative co-efficient, and r the number of terms preceding the first negative term. To take the most unfavorable case, suppose all the terms after the first negative term to be negative, and each negative co-efficent to be $-g$. Then the equation becomes

$$(1) \quad x^n + ax^{n-1} + \ldots - g(x^{n-r} + \ldots + x + 1) = 0.$$

Since the series within the parenthesis is a geometrical progression, (1) becomes

$$(2) \quad x^n + ax^{n-1} + \ldots - \frac{g(x^{n-r+1} - 1)}{x-1} = 0.$$

Dropping all the positive terms in (2) but x^n, we have

$$(3) \quad x^n - \frac{g(x^{n-r+1} - 1)}{x-1} = 0.$$

Now, a superior limit of x in (3) will, for a stronger reason, be a superior limit in (2), since the first member of (2) is the same as that of (3), with the addition of positive terms. But the superior limit of x in (3) will be found if we find the superior limit in

$$(4) \quad x^n - \frac{gx^{n-r+1}}{x-1} = 0.$$

For the positive parts of (3) and (4) are the same, and the negative part of (3) is less, numerically, than that of (4).

Clearing (4) of fractions, dividing by x^{n-r+1}, and transposing g, we have

(5) $x^{r-1}(x-1)=g.$

Since $x-1<x$, $(x-1)^{r-1}<x^{r-1}$, $\therefore$ $(x-1)^r<x^{r-1}(x-1)$. Therefore, a limit of x in (5) will be found if we find the value of x in

(6) $(x-1)^r=g$, $\therefore$ $x-1=\sqrt[r]{g}$, $\therefore$ $x=1+\sqrt[r]{g}$.

Therefore, $1+\sqrt[r]{g}$ is a superior limit of the positive roots.

435. Proposition.

The inferior limit of the positive roots is the reciprocal of the superior limit of the positive roots of the transformed equation, obtained by substituting $\frac{1}{y}$ *for* x *in the given equation.*

436. Proposition.

The limits of the negative roots are the limits, with their signs changed, of the positive roots of the transformed equation, obtained by substituting $-$ y *for* x *in the given equation.*

437. Example.

Find the positive and negative limits of x in the equation

$$x^4+4x^3-x^2-16x-12=0.$$

Ans. $\begin{cases}\text{Positive limits, 5 and .6.}\\ \text{Negative limits, } -13 \text{ and } -\frac{3}{7}.\end{cases}$

438. Proposition.

If the real roots of an equation, taken in the descending order of their magnitudes, be r, s, t, . . . , *then, if the quantities of another descending series,* r′, s′, t′, . . . , *in which* r′ > r, r > s′ > s, s > t′ > t, . . . , *be substituted for* x *in the equation, the results will be alternately positive and negative.*

By factoring, the equation becomes

$$(1) \quad (x-r)(x-s)(x-t)\ldots=0.$$

Substituting r', s', t', . . . , in succession for x in (1), we have

$$(2) \quad (r'-r)(r'-s)(r'-t)\ldots>0,$$

since the factors are all positive.

$$(3) \quad (s'-r)(s'-s)(s'-t)\ldots<0,$$

since one factor is negative and the rest positive.

$$(4) \quad (t'-r)(t'-s)(t'-t)\ldots>0,$$

since two factors are negative and the rest positive.

Cor. If, when two numbers are substituted for x, the results have unlike signs, there is an odd number of roots included between those numbers; but if the results have like signs, there is no root or an even number of roots included between them.

439. Problem.

To find the integral roots of an equation.

Find, by Art. 385, the value of the first member when the integral divisors of the last term which are included within the limits of the roots are substituted for the unknown quantity. Those divisors which reduce the first member to 0 are roots.

440. Examples.

1. $x^4 + 4x^3 - x^2 - 16x - 12 = 0.$

Ans. $2, -1, -2, -3.$

2. $x^3 - 9x^2 + 23x - 15 = 0.$ *Ans.* $1, 3, 5.$

3. $x^3 - 3x^2 - 10x + 24 = 0.$ *Ans.* $2, 4, -3.$

4. $x^4 - 13x^2 + 36 = 0.$ *Ans.* $2, 3, -2, -3.$

5. $x^5 - x^4 - 13x^3 + 13x^2 + 36x - 36 = 0.$

Ans. $1, 2, 3, -2, -3.$

6. $x^4 - 2x^3 - 11x^2 + 42x - 40 = 0.$

Ans. $2, -4, 2 + \sqrt{-1}, 2 - \sqrt{-1}.$

STURM'S THEOREM.

441. Statement.

Let $f(x) = 0$ *be an equation of the* n^{th} *degree freed from equal roots; let* $f_1(x)$ *be its first derived function; let the operation for finding the g. c. d. be applied to* $f(x)$ *and* $f_1(x)$, *with the modification that the signs of the remainders be changed; let the operation be continued till a remainder is obtained which is independent of* x; *let the modified remainders be denoted by* $f_2(x)$, $f_3(x), \ldots f_r(x), \ldots f_n(x)$; *then, if in the series* $f(x)$, $f_1(x)$, $f_2(x), \ldots f_r(x), \ldots f_n(x)$, *we make* $x = p$, *and arrange the signs of the results in a line, and if we make* $x = v$, *a number algebraically greater than* p, *and arrange the signs of the results in a line, the number of variations of signs in the first arrangement minus the number of variations in the second arrangement will be equal to the number of the real roots of* $f(x) = 0$ *comprehended between* p *and* v.

442. Preliminary Relations.

Call $f(x), f_1(x), f_2(x), \ldots f_r(x), \ldots f_n(x)$ Sturm's functions; and call $f_1(x), f_2(x), \ldots f_r(x), \ldots f_n(x)$ auxiliary functions.

Let $q_1, q_2, q_3, \ldots q_r, \ldots q_{n-1}$ be the quotients obtained in applying the process for finding the g. c. d. to $f(x)$ and $f_1(x)$. Then we shall have

$$f(x) = q_1 f_1(x) - f_2(x).$$

$$f_1(x) = q_2 f_2(x) - f_3(x).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{n-2}(x) = q_{n-1} f_{n-1}(x) - f_n(x).$$

443. Lemmas.

1. $f_n(x)$ is not 0; for, by hypothesis, it is independent of x, and if it were 0, then $f(x)$ and $f_1(x)$ would have a common divisor involving x, and $f(x)$ would have equal roots; but, by hypothesis, $f(x)$ has been freed from equal roots; hence, $f_n(x)$ is not 0.

2. No two consecutive functions can reduce to 0 for the same value of x; for suppose that a certain value substituted for x should make, for example, $f_1(x) = 0$ and $f_2(x) = 0$; then, since $f_1(x) = q_2 f_2(x) - f_3(x)$, we shall have $f_3(x) = 0$; then, since $f_2(x) = q_3 f_3(x) - f_4(x)$, we shall have $f_4(x) = 0$, and so on, till we should find $f_n(x) = 0$, which is impossible by Lemma 1; hence, no two consecutive functions can reduce to 0 for the same value of x.

3. If any of the functions reduce to 0 when a certain value is substituted for x, the function preceding and the function following will have unlike signs for the same value.

for, if a certain value substituted for x makes $f_r(x)=0$, then $f_{r-1}(x)=-f_{r+1}(x)$; that is, $f_{r-1}(x)$ and $f_{r+1}(x)$ have unlike signs.

4. As x increases from p toward v, the number of variations will be neither increased nor diminished till x passes through a root of $f(x)=0$, when a variation will be lost. There will be no change of sign in any of Sturm's functions, except when x passes through a root of that function. Let q be a root of $f_r(x)$; then $f_r(q)=0$, and $f_{r-1}(q)$ and $f_{r+1}(q)$ will have contrary signs. Then, just before and just after $x=q$ the three functions, $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$, will give one permanence and one variation of signs, so that no variation will be either lost or gained in passing through a root of any of the auxiliary functions.

Now, let r be a root of $f(x)=0$, and let $r-h$ and $r+h$ be substituted for x, and the results be developed and arranged according to the ascending powers of h; then, since $f(r)=0$ and $f_1(r)=f'(r)$, we have, Art. 395,

$$f(r-h)=f_1(r)(-h)+f''(r)\frac{(-h)^2}{\underline{|2}}+f'''(r)\frac{(-h)^3}{\underline{|3}}+\dots$$

$$f(r+h)=f_1(r)\,h+f''(r)\frac{h^2}{\underline{|2}}+f'''(r)\frac{h^3}{\underline{|3}}+\dots$$

Now, so small a value may be assigned to h as to make the term containing the first power of h greater than the sum of all the following terms; hence, the sign of $f(r-h)$ will be the same as that of $f_1(r)(-h)$, and the sign of $f(r+h)$ will be the same as that of $f_1(r)\,h$.

Since r is not a root of $f_1(x)=0$, h may be taken so small that no root of $f_1(x)=0$ shall lie between $r-h$ and $r+h$; then the signs of $f_1(r-h)$, $f_1(r)$, $f_1(r+h)$ will be the same.

But the sign $f(r-h)$ is like the sign of $f_1(r)(-h)$, or unlike the sign of $f_1(r)$, since h is negative; but the sign of $f_1(r-h)$ is like the sign of $f_1(r)$; $\therefore$ $f(r-h)$ and $f_1(r-h)$ have unlike signs. Again, the sign of $f(r+h)$ is like the sign of $f_1(r)\,h$, or like the sign of $f_1(r)$, since h is positive; but the sign of $f_1(r+h)$ is like the sign of $f_1(r)$; $\therefore$ $f(r+h)$ and $f_1(r+h)$ have like signs. Therefore, in passing from $r-h$ to $r+h$, Sturm's functions lose one variation of signs.

Now, since Sturm's functions never lose or gain a variation of signs except when x passes through a root of $f(x)=0$, when a variation is always lost, the number of variations lost in passing from p to v will denote the number of real roots between p and v.

Cor. 1. The number of variations lost in passing from $-\infty$ to $+\infty$ will be the number of real roots of the equation, since $-\infty$ and $+\infty$ are limits of the roots. In substituting $-\infty$ and $+\infty$ for x in Sturm's functions, it will be sufficient to determine the sign of the first term, since this term will become greater than the sum of all the other terms, and will determine the sign of the result.

Cor. 2. In passing any root of $f(x)=0$ to the root next in order, we must pass a root of $f_1(x)=0$; for, just after passing any root of $f(x)=0$, $f(x)$ and $f_1(x)$ have like signs, and just before reaching the next root of $f(x)=0$, $f(x)$ and $f_1(x)$ have unlike signs; but since the sign of $f(x)$ has not changed, the sign of $f_1(x)$ has changed; hence, a root of $f_1(x)=0$ has been passed; hence, between any two consecutive roots of $f(x)=0$ there is a root of $f_1(x)=0$.

Cor. 3. The limits of the roots can be found by substituting 0, 1, 2, 3, ... in Sturm's functions, till a number is found which gives the same number of variations as $+\infty$; and by substituting 0, -1, -2, -3, ... till a number is

found which gives the same number of variations as $-\infty$. The situation of the roots can be found by substituting $0, 1, 2, 3, \ldots, -1, -2, -3, \ldots$, and observing when a variation is lost.

444. Application.

Find the limits and situation of the roots of

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

OPERATION.

$$f(x) = x^4 - 8x^3 + 14x^2 + 4x - 8.$$
$$f_1(x) = x^3 - 6x^2 + 7x + 1.$$
$$f_2(x) = 5x^2 - 17x + 6.$$
$$f_3(x) = 76x - 103.$$
$$f_4(x) = 15125.$$

In deriving $f_1(x)$, $f_2(x)$, ..., reject positive factors.

	$f(x)$,	$f_1(x)$,	$f_2(x)$,	$f_3(x)$,	$f_4(x)$.	
$x = -\infty$	+	−	+	−	+	4 variations.
$x = +\infty$	+	+	+	+	+	0 variation.
$x = 0$	−	+	+	−	+	3 variations.
$x = 1$	+	+	−	−	+	2 variations.
$x = 2$	+	−	−	+	+	2 variations.
$x = 3$	−	−	±	+	+	1 variation.
$x = 4$	−	−	+	+	+	1 variation.
$x = 5$	−	+	+	+	+	1 variation.
$x = 6$	+	+	+	+	+	0 variation.
$x = -1$	+	−	+	−	+	4 variations.

Since $+6$ gives the same number of variations as $+\infty$, and -1 the same number as $-\infty$, $+6$ and -1 are limits of the roots.

Since 4 variations are lost in passing from $-\infty$ to $+\infty$, the equation has 4 real roots.

Since 3 variations are lost in passing from 0 to $+6$, the equation has 3 positive roots, and hence one negative root.

Since 1 variation is lost in passing from 0 to $+1$, one root lies between 0 and $+1$; $\therefore$ this root $= 0 +$ a fraction.

Since 1 variation is lost in passing from $+2$ to $+3$, one root lies between $+2$ and $+3$; $\therefore$ this root $= 2 +$ a fraction.

Since 1 variation is lost in passing from $+5$ to $+6$, one root lies between $+5$ and $+6$; $\therefore$ this root $= 5 +$ a fraction.

Since 1 variation is lost in passing from -1 to 0, one root lies between -1 and 0; $\therefore$ this root $=$ a negative fraction.

If two or more variations should be lost in passing from one integral number to the next consecutive number, the roots lying between these numbers can be separated by increasing the less of these numbers by .1, .2, .3, . . . , or, if necessary, by .01, .02, .03, . . . , till a variation is lost.

445. Examples.

1. Find the integral part of the roots of the equation $x^3 - 7x + 7 = 0$. *Ans.* 1, 1, -3.

2. Find the integral part of the real roots of the equation $x^4 + x^3 + x^2 + 3x - 100 = 0$. *Ans.* 2, -3.

3. Find the integral part of the roots of the equation $x^4 - 12x^2 + 12x - 3 = 0$. *Ans.* 0, 0, 2, -3.

4. Find the integral part of the real root of the equation $x^3 + 2x^2 - 3x = 9$. *Ans.* 1.

5. Discuss $x^5 - 7x^4 + 13x^3 + x^2 - 16x + 4 = 0$.

   Ans. The roots are 2, 2, -1, $2 + \sqrt{3}$, $2 - \sqrt{3}$.

HORNER'S METHOD OF APPROXIMATION.

446. General Statement.

1. The integral parts of the roots are first found by Sturm's Theorem.

2. The equation is then transformed into another whose roots are less than the roots of the given equation by the integral part of a certain root of the given equation. One root, at least, of the transformed equation will then be the decimal part of that root of the given equation.

3. The first figure of this decimal root is found by substituting .1, .2, .3, . . . in the transformed equation, till two successive substitutions give results with contrary signs, in which case the less will be the tenths of the root.

4. This transformed equation is then transformed into another whose roots are less than the roots of the first transformed equation by the tenths just obtained. One root of this second transformed equation will be the remaining part of the root of the first transformed equation, and so on.

447. Illustration.

Let us find one root of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

By Sturm's Theorem we find that 5 is the integral part of one root.

We next transform the equation into another whose roots are less than the roots of this equation by 5, thus:

$$
\begin{array}{rrrrr}
1 & -\ 8 & +\ 14 & +\ \ 4 & -\ \ 8 \\
 & +\ 5 & -\ 15 & -\ \ 5 & -\ \ 5 \\
\cline{2-5}
 & -\ 3 & -\ \ 1 & -\ \ 1 & -\ 13 \\
 & +\ 5 & +\ 10 & +\ 45 & \\
\cline{2-4}
 & +\ 2 & +\ \ 9 & +\ 44 & \\
 & +\ 5 & +\ 35 & & \\
\cline{2-3}
 & +\ 7 & +\ 44 & & \\
 & +\ 5 & & & \\
\cline{2-2}
 & +\ 12 & & &
\end{array}
$$

The transformed equation then is

$$y^4 + 12y^3 + 44y^2 + 44y - 13 = 0.$$

Let us now substitute .1, .2, .3, . . . , in succession, in this equation till two successive substitutions give results with contrary signs, thus:

$$
\begin{array}{rrrrr}
1 & +\ 12 & +\ 44 & +\ 44 & -\ 13 \\
 & +\ \ .1 & +\ \ 1.21 & +\ \ 4.521 & +\ \ 4.8521 \\
\cline{2-5}
 & +\ 12.1 & +\ 45.21 & +\ 48.521 & -\ \ 8.1479
\end{array}
$$

The result of this substitution is -8.1479.

$$
\begin{array}{rrrrr}
1 & +\ 12 & +\ 44 & +\ 44 & -\ 13 \\
 & +\ \ .2 & +\ \ 2.44 & +\ \ 9.288 & +\ 10.6576 \\
\cline{2-5}
 & +\ 12.2 & +\ 46.44 & +\ 53.288 & -\ \ 2.3424
\end{array}
$$

The result of this substitution is -2.3424.

$$
\begin{array}{rrrrr}
1 & +\ 12 & +\ 44 & +\ 44 & -\ 13 \\
 & +\ \ .3 & +\ \ 3.69 & +\ 14.307 & +\ 17.4921 \\
\cline{2-5}
 & +\ 12.3 & +\ 47.69 & +\ 58.307 & +\ \ 4.4921
\end{array}
$$

The result of this substitution is $+4.4921$.

Since .2 and .3 give results with contrary signs, one root is between .2 and .3, or .2 is the tenths of the root.

We next transform the first transformed equation into another whose roots are less than the roots of this equation by .2, thus:

$$\begin{array}{rrrrr}
1 & +12 & +44 & +44 & -13 \\
 & +\ \ .2 & +\ \ 2.44 & +\ \ 9.288 & +10.6576 \\
\hline
 & +12.2 & +46.44 & +53.288 & -\ \ 2.3424 \\
 & +\ \ .2 & +\ \ 2.48 & +\ \ 9.784 & \\
\cline{2-4}
 & +12.4 & +48.92 & +63.072 & \\
 & +\ \ .2 & +\ \ 2.52 & & \\
\cline{2-3}
 & +12.6 & +51.44 & & \\
 & +\ \ .2 & & & \\
\cline{2-2}
 & +12.8 & & &
\end{array}$$

Therefore, the second transformed equation is

$$z^4 + 12.8z^3 + 51.44z^2 + 63.072z - 2.3424 = 0.$$

Since z is a small fraction, we can determine the hundredths by neglecting, for the present, the terms containing powers of z higher than the first. We shall then have

$$63.072z - 2.3424 = 0; \ \therefore \ z = .03 +,$$

which is the hundredths.

We next transform the second transformed equation into another whose roots are less than the roots of this equation by .03, and so on.

These successive transformations may be combined as in the following condensed form:

<table>
<tr><td>1 −8</td><td>+ 14</td><td>+ 4</td><td>− 8 (5.2360679</td></tr>
<tr><td>5</td><td>− 15</td><td>− 5</td><td>− 5</td></tr>
<tr><td>−3</td><td>− 1</td><td>− 1</td><td>−13*</td></tr>
<tr><td>5</td><td>10</td><td>+ 45</td><td>10.6576</td></tr>
<tr><td>2</td><td>9</td><td>44*</td><td>− 2.3424*</td></tr>
<tr><td>5</td><td>35</td><td>9.288</td><td>1.93880241</td></tr>
<tr><td>7</td><td>44*</td><td>53.288</td><td>− .40359759*</td></tr>
<tr><td>5</td><td>2.44</td><td>9.784</td><td>.39905490</td></tr>
<tr><td>12*</td><td>46.44</td><td>63.072*</td><td>− .00454269</td></tr>
<tr><td>.2</td><td>2.48</td><td>1.554747</td><td>.00400954</td></tr>
<tr><td>12.2</td><td>48.92</td><td>64.626747</td><td>− .00053315</td></tr>
<tr><td>.2</td><td>2.52</td><td>1.566321</td><td>.00046777</td></tr>
<tr><td>12.4</td><td>51.44*</td><td>66.19306|8*</td><td>− .00006538</td></tr>
<tr><td>.2</td><td>.3849</td><td>.31608|</td><td>.00006014</td></tr>
<tr><td>12.6</td><td>51.8249</td><td>66.50915</td><td>.00000524</td></tr>
<tr><td>.2</td><td>.3858</td><td>.31656</td><td></td></tr>
<tr><td>12.8*</td><td>52.2107</td><td>66.825|71</td><td></td></tr>
<tr><td>.03</td><td>.3867</td><td></td><td></td></tr>
<tr><td>12.83</td><td>52.59|74*</td><td></td><td></td></tr>
<tr><td>.03</td><td>.08|</td><td></td><td></td></tr>
<tr><td>12.86</td><td>52.68</td><td></td><td></td></tr>
<tr><td>.03</td><td>.08</td><td></td><td></td></tr>
<tr><td>12.89</td><td>52.76</td><td></td><td></td></tr>
<tr><td>.03</td><td></td><td></td><td></td></tr>
<tr><td>1|2.92*</td><td></td><td></td><td></td></tr>
</table>

After obtaining two decimal places, the work can be contracted by omitting unnecessary decimal places at the right; and after three decimal places have been found, the remaining figures of the root can be found with sufficient

accuracy by division. The other positive roots can be found in a similar manner. The negative roots can be found by substituting $-y$ for x, and finding the positive roots of the resulting equation.

448. Rule.

1. *Find the integral parts of the roots, by Sturm's Theorem.*

2. *Transform the equation into another whose roots are less than the roots of the given equation by the integral part of one of the positive roots.*

3. *Substitute* .1, .2, .3, . . . *in the transformed equation till two successive substitutions give results with contrary signs, and the less will be the tenths of the root.*

4. *Transform this transformed equation into another whose roots are less than the roots of this equation by the tenths just obtained.*

5. *Disregard all the terms containing higher powers of the unknown quantity than the first, and find the hundredths from the resulting equation.*

6. *Transform the second transformed equation into another whose roots are less than the roots of this equation by the hundredths just obtained.*

7. *Disregard all the terms containing higher powers of the unknown quantity than the first, and find from the resulting equation the remaining figures of the root.*

449. Examples.

1. Given $x^3 - 7x + 7 = 0$, to find x.

Ans. 1.356896, 1.692021, $-$ 3.048917.

2. Given $x^4 - 12x^2 + 12x - 3 = 0$, to find x.

Ans. 2.858083, .6060183, .443278, $-$ 3.907378.

3. Given $x^3 - 3x^2 - 2x + 5 = 0$, to find x.
Ans. 3.128419, 1.201639, -1.330059.

4. Given $x^3 = 41063625$, to find x. *Ans.* 345.

5. Extract the cube root of 43614208. *Ans.* 352.

6. Extract the fifth root of 36936242722357.
Ans. 517.

CUBIC EQUATIONS.

450. Cardan's Formula.

Causing the second term to disappear, the cubic equation will take the form

$$(1) \quad x^3 + bx + c = 0.$$

Let $y + z = x$; then we shall have

$$(2) \quad (y+z)^3 + b(y+z) + c = 0.$$

$$\therefore \quad (3) \quad y^3 + z^3 + (3yz + b)(y + z) + c = 0.$$

Let $3yz + b = 0$; then $z = -\dfrac{b}{3y}$, and (3) becomes

$$(4) \quad y^3 + z^3 + c = 0.$$

Substituting the value of z, (4) becomes

$$(5) \quad y^3 - \frac{b^3}{27y^3} + c = 0.$$

$$\therefore \quad (6) \quad y^6 + cy^3 = \frac{b^3}{27}.$$

$$\therefore \quad (7) \quad y^3 = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}.$$

$$\therefore \quad (8) \quad y = \sqrt[3]{-\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}.$$

Substituting the value of y^3 in (4), we find

$$(9)\quad z^3 = -\frac{c}{2} \mp \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}.$$

$$\therefore\quad (10)\quad z = \sqrt[3]{-\frac{c}{2} \mp \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}.$$

But $x = y + z$; hence, we shall have

$$(11)\quad x = \sqrt[3]{-\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{-\frac{c}{2} \mp \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}.$$

If $\frac{c^2}{4} + \frac{b^3}{27}$ is negative, the formula fails, since the root can not be obtained. In this case, contrary to appearance, the roots are all real and unequal, as is thus proved:

$$\begin{aligned} f\,(x) &= \quad x^3 + bx + c. \\ f_1(x) &= \quad 3x^2 + b. \\ f_2(x) &= -\,2bx - 3c. \\ f_3(x) &= -\,4b^3 - 27c^2. \end{aligned}$$

In order that the roots be all real, $-\infty$ substituted in Sturm's functions must give three variations, and $+\infty$ three permanences, which can be the case only when

$$-4b^3 - 27c^2 > 0.$$

$$\therefore\qquad \frac{c^2}{4} + \frac{b^3}{27} < 0.$$

In this case b is negative, and $\frac{b^3}{27} > \frac{c^2}{4}$. The roots are unequal, since $f_3(x)$ is not 0.

By combining the signs in Cardan's formula, nine values are obtained; but a cubic equation can have but three roots, and these roots must satisfy the condition $yz = -\frac{b}{3}$.

451. Examples.

1. Given $x^3 - 12x - 16 = 0$, to find x.

Ans. $4, -2, -2$.

2. Given $x^3 + x^2 - 8x = 12$, to find x.

Ans. $3, -2, -2$.

3. Given $x^3 + 6x^2 - 32 = 0$, to find x.

Ans. $2, -4, -4$.

4. Given $x^3 - 27x + 54 = 0$, to find x.

Ans. $3, 3, -6$.

5. Given $x^3 + 3x^2 - 24x - 80 = 0$, to find x.

Ans. $5, -4, -4$.

6. Given $x^3 - 6x^2 + 13x - 10 = 0$, to find x.

Ans. $2, 2 + \sqrt{-1}, 2 - \sqrt{-1}$.

452. Evans's Solution of Cubics.

(1) $x^3 = ax + b$.

$\therefore$ (2) $x^2 = a + \frac{b}{x}$.

$\therefore$ (3) $x = \pm\sqrt{a + \frac{b}{x}}$.

$\therefore$ (4) $$x = \pm\sqrt{a + \cfrac{b}{\pm\sqrt{a + \cfrac{b}{\pm\sqrt{a + \cfrac{b}{\pm\sqrt{a + \cfrac{b}{\vdots}}}}}}}}$$

453. Rule.

1. *Find the square root of* a + b, *divide* b *by this root, add* a *to the quotient, extract the square root of the sum, divide* b *by this root, and so on; the successive roots will afford nearer and nearer approximations to the true root of the equation.*

2. *If* b *is positive, there will be one positive root, which may be found by considering the radicals all positive.*

3. *If* b *is negative, there will be one negative root, which may be found by considering the radicals all negative.*

4. *To find the other roots, transpose all the terms of the equation to the first member, which divide by the unknown quantity minus the root obtained, and the quotient placed equal to* 0 *will be a quadratic which will give the other roots.*

Scholium. This method is applicable to the failing case of Cardan's formula, and Cardan's formula is applicable to the failing case of Evans's method.

454. Examples.

1. Given $x^3 = 6x + 2$, to find x.
Ans. 2.601679, —.339877, —2.261802.

2. Given $x^3 - 7x = -7$, to find x. *Ans.* —3.048.

BIQUADRATIC EQUATIONS.

455. Descartes's Method.

Causing the second term to disappear, the biquadratic equation will take the form

$$(1) \quad x^4 + bx^2 + cx + d = 0.$$

Assuming the first member resolved into quadratic factors, the co-efficients of x will be equal with contrary signs, since there is no term involving x^3, and we shall have

(2) $x^4 + bx^2 + cx + d = (x^2 + mx + n)(x^2 - mx + p)$.

We are now to find values for m, n, and p which will satisfy the equation. Developing, we have

(3) $x^4 + bx^2 + cx + d = x^4 + (p + n - m^2)x^2 + m(p - n)x + pn$.

Equating the co-efficients of the like powers of x, we have

(4) $p + n - m^2 = b$.

(5) $m(p - n) = c$.

(6) $pn = d$.

(4) gives (7) $p + n = m^2 + b$.

(5) gives (8) $p - n = \frac{c}{m}$.

$\frac{(7) + (8)}{2} =$ (9) $p = \frac{1}{2}\left(m^2 + b + \frac{c}{m}\right)$.

$\frac{(7) - (8)}{2} =$ (10) $n = \frac{1}{2}\left(m^2 + b - \frac{c}{m}\right)$.

Substituting the values of p and n in (6), and reducing, we shall have

(11) $m^6 + 2bm^4 + (b^2 - 4d)m^2 - c^2 = 0$.

Taking m^2 as the unknown quantity, this equation may be regarded as a cubic. It will have one real root, since the last term is negative; and this root may be found by Horner's method, Cardan's formula, or Evans's method. We may therefore regard m^2, and hence m, as known.

Now, equation (1) will be satisfied

$$\text{If } \left\{\begin{array}{l} x^2+mx+n=0. \\ x^2-mx+p=0. \end{array}\right\} \therefore \left\{\begin{array}{l} x=\dfrac{-m\pm\sqrt{m^2-4n}}{2}. \\ x=\dfrac{m\pm\sqrt{m^2-4p}}{2}. \end{array}\right.$$

456. Examples.

1. Given $x^4+4x^3+3x^2-44x-84=0$, to find x.

Ans. $3, -2, \dfrac{-5\pm\sqrt{-31}}{2}$.

2. Given $x^4-6x^2-8x-3=0$, to find x.

Ans. $-1, -1, -1, 3$.

3. Given $x^4-12x^3+49x^2-78x+40=0$, to find x.

Ans. $1, 2, 4, 5$.

4. Given $\left\{\begin{array}{l} x^2+y=7. \\ y^2+x=11. \end{array}\right\}$ Required x and y.

Ans. $\left\{\begin{array}{l} x=2,\ 3.131313,\ -1.848126, \text{ or } -3.283186. \\ y=3,\ -2.805118,\ 3.584428, \text{ or } -3.779309. \end{array}\right.$

Two equations of the m^{th} and n^{th} degree respectively, will, by elimination, give one equation of the mn^{th} degree.